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Extending Fully Fuzzy Sylvester Matrix Equations to Neutrosophic Environments: An Improved Solution Method

Myra Suphelea Asmizal[1], Wan Suhana Wan Daud[1,*], Ghassan Malkawi[2], Khairu Azlan Abd Aziz[3]

[1] Department of Mathematical Sciences, Faculty of Intelligent Computing, Universiti Malaysia Perlis, 02600, Arau, Perlis, Malaysia
[2] Department of Computer Information Science, Higher Colleges of Technology, United Arab Emirates
[3] Universiti Teknologi Mara, Perlis Branch, Arau Campus, 02600, Perlis, Malaysia

ABSTRACT

The classical Sylvester matrix equations play a fundamental role in system analysis, model transformation and controller design. However, there are limits in handling uncertainty information as all parameters of the system are assumed to be precisely known. Uncertainty can occur in many real world applications so the classical mathematical models are insufficient to represent the system behaviour accurately. Even fuzzy Sylvester equations are unable to capture indeterminacy or contradiction within the data. Therefore, the study of solution framework for Fully Fuzzy Neutrosophic Sylvester Matrix Equation (FFNSME) that incorporates truth, indeterminacy and falsity components is introduced to overcome these limitations. Left– right triangular neutrosophic fuzzy numbers are used to represent uncertain parameters within the matrices. The Associated Linear Systems approach and the score function method are applied for deneutrosophication to enable computational processing while preserving embedded uncertainty. The findings demonstrate that the FFNSME framework provides a flexible representation of uncertainties compared to classical or fuzzy Sylvester equations. This enhances the reliability of system modeling, controller synthesis and matrix equation solutions when working with vague, contradictory or incomplete information. The formulation is applicable to a wide range of linear and nonlinear control problems. Future research may explore optimized numerical algorithms and extensions to large-scale, highdimensional and time-varying systems to further improve computational efficiency.

Keywords: Homotopy Analysis Metod; tangent vector; solution path; nonlinearity; differential equation; Euler; Newton; predictor corrector; global quadratic convergence

1. Introduction

An expression of the mathematical relationships between variables and constants in matrix form is called matrix equation. These equations are extensively applied in various fields for modelling and solving problems, such as in physics, engineering, computer science and control systems. The solution of matrix equations commonly involves the matrix addition, multiplication and inversion [1]. Morever, an efficient and structured framework is offered through matrix equations for representing and computing systems of linear equations, linear transformations and other operations involving numerical arrays [2].

Matrix equations, typically the Sylvester matrix equation, are widely used in mathematical modelling which are appearing in both linear and nonlinear forms [3]. The Sylvester equation plays a

* *Corresponding author.*
E-mail address: wsuhana@unimap.edu.my

fundamental role in the stability analysis of control systems, particularly in assessing the stability of linear time- invariant (LTI) systems. The solutions can be obtained using direct or iterative numerical methods. The Bartels-Stewart algorithm as one of the examples of direct methods computes exact solutions by transforming the system into a simpler equivalent form [4], whereas iterative method such as Krylov subspace approaches are more efficient for large-scale problems due to their lower computational complexity [5].

The applicability of classical methods becomes limited as many real-world applications involve uncertainty and incomplete information. To address this issue, Zadel *et al.,* [6] introduced fuzzy sets which allow partial membership values in the interval [0, 1] and provide a framework for modelling imprecise data. Several extensions, including interval-valued fuzzy sets (IVFS) [7], cubic sets [8], soft sets [9] and cubic sets [10], were developed to enhance uncertainty representation. However, fuzzy models still not sufficient to handle indeterminate information. To overcome the limitation, Smarandache *et al.,* [11] proposed neutrosophic sets, characterized by three membership functions which are truth (*T*), indeterminacy (*I*) and falsity (*F*). Overall, this framework provides greater flexibility in modelling complex uncertainty and has been applied in diverse fields such as decision-making [17], image processing [28] and pattern recognition [29]. A comparative study conducted by Asmizal *et al*., [9] indicates that neutrosophic numbers provide a more effective environment for solving uncertain matrix equations than classical fuzzy models.

Hence, this study is aimed to extend the existing fuzzy Sylvester framework by formulating an improved solution method for the Fully Fuzzy Neutrosophic Sylvester Matrix Equation (FFNSME) of

$$\tilde{A}\tilde{X}+\tilde{X}\tilde{B}=\tilde{C} \tag{1}$$

where the coefficients $\tilde{A},\tilde{B},\tilde{C}$ and the solution $\tilde{X}$ are in the form of left right-triangular neutrosophic fuzzy numbers (LR-TriNFN) of $(m,\alpha,\beta;T,I,F)$.

The significance of this research lies in its ability to bridge the gap between theoretical matrix algebra and real-world uncertainty. By extending the Fully Fuzzy Sylvester Matrix Equation into neutrosophic environments, this study provides a better mathematical framework that considers the truth, falsity, and indeterminacy. Furthermore, the improved solution method developed a practical computational technique that handle uncertain data without losing the important characteristics. Consequently, this study offers a vital tool for researchers in control theory and system modeling, allowing more reliable simulations in fields where data is often incomplete or contradictory.

The remaining part of this paper is structured as follows. Section 2 presents some preliminaries that discuss the foundational concepts of fuzzy numbers and neutrosophic numbers with their mathematical properties. Section 3 presents the methodology with the theoretical formulation of the FFNSME, meanwhile section 4 provides an illustrative example. Lastly, section 5 concludes the study with recommendations for future research.

2. Preliminaries

This section will recall some fundamental concepts including definitions and theorems related to this study.

2.1 Theory of Fuzzy Numbers

The definitions of fuzzy numbers are given as follows.

Definition 1. [5] Let X be a nonempty set, the fuzzy set $\tilde{A}$ in X is characterized by its membership function,

$$\mu_{\tilde{A}} : X \rightarrow [0,1] \tag{2}$$

and $\mu_{\tilde{A}}(x)$ represents the degree of membership of the element x in fuzzy numbers $\tilde{A}$ for each $x \in X$.

Definition 2. [5] The fuzzy set $\tilde{A}$ is represented by a set of ordered pairs of elements x and $\mu_{\tilde{A}}$ which can be written as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}. \tag{3}$$

Definition 3. [5] A fuzzy set $\tilde{A}$ in X has the following properties as described below,

1) $\mu_{\tilde{A}}$ is upper semicontinuous.
2) $\mu_{\tilde{A}} = 0$ is outside of some interval $[c,d]$.
3) There are real numbers of a and b, such that $c \le a \le d$ and
   (a) $\mu_{\tilde{A}}$ is monotonic increasing on $[c,a]$,
   (b) $\mu_{\tilde{A}}$ is monotonic decreasing on $[b,d]$,
   (c) $\mu_{\tilde{A}} = 1$ for $a \le x \le b$.

Various types of fuzzy numbers have been extensively discussed in the literature, with triangular fuzzy numbers (TriFNs) being among the most commonly used. One particular form of TriFN is the LR-triangular fuzzy number (LR-TriFN). For more detail, the definition of LR-TriFN is provided below.

Definition 4. [12] The LR-TriFN $\tilde{M} = (m, \alpha, \beta)$ has membership function as follows,

$$\tilde{\mu}_{\tilde{M}} = \begin{cases} 1 - \dfrac{m-x}{\alpha}, & m-\alpha \le x \le m \\ 1 - \dfrac{x-m}{\beta}, & m \le x \le m+\beta \\ 0, & otherwise, \end{cases} \tag{4}$$

where m is the mean value while α and β are the spreads on the left and right sides, respectively.

Definition 5. [30] A min-max function is a collection of equations such that at least one of the equations has a min or max equation. Consider the positive two fuzzy numbers of $\tilde{A} = (m, \alpha, \beta)$ and $\tilde{B} = (n, \gamma, \delta)$, if $\tilde{A}$ is positive, then the following inequalities are satisfied:

$$0 \le (m-\alpha)(n-\gamma) \le (m-\alpha)(n+\delta), \tag{5}$$
$$0 \le (m+\beta)(n-\gamma) \le (m+\beta)(n+\delta), \tag{6}$$

Definition 6. [13] The product of two positive fuzzy numbers $\tilde{A}=(m,\alpha,\beta)$ and $\tilde{B}=(n,\gamma,\delta)$ is defined as

$$\tilde{A}\otimes\tilde{B}=(mn,f_1,f_2), \tag{7}$$

where

$$f_1=mn-Min((m-\alpha)(n-\gamma),(m+\beta)(n-\gamma)),$$
$$f_2=Max((m-\alpha)(n+\gamma),(m+\beta)(n+\gamma))-mn.$$

Nevertheless, this method remains limited in handling incomplete or inconsistent data effectively. Hence, neutrosophic fuzzy numbers (NFN) were developed as an extension of both fuzzy numbers. The next subsection will explain more detail about the theory of neutrosophic numbers.

2.2 Theory of Neutrosophic Numbers

Neutrosophic fuzzy numbers (NFN) offered a more flexible framework than traditional fuzzy and intuitionistic fuzzy numbers as not only the incomplete and uncertain information can be managed but also the indeterminate data. This makes the neutrosophic numbers more effective for handling complex real-world problems. The definitions of neutrosophic numbers are as follows:

Definition 7. [14] Let X be a universal set and let $x\in X$. The neutrosophic set A in X is characterized by a truth (*T*), indeterminacy (*I*) and falsity (*F*) membership functions, then neutrosophic numbers is given as,

$$A=\{(x,T(x),I(x),F(x)\mid x\in X\}. \tag{8}$$

There is a restriction on the sum of T, I and F which must be satisfied, which is

$$0+\leq T+I+F\leq 3+. \tag{9}$$

In terms of philosophy, the neutrosophic set is derived from actual standard or non-standard subsets of $]0-,1+[$. However, for technical applications the interval [*0,1*] is more appropriate to be used [15]. Same goes to fuzzy numbers, neutrosophic fuzzy numbers also classify into triangular neutrosophic fuzzy numbers (TriNFN). For instance, a TriNFN is defined by three parameters representing the lower, middle, and upper values, each associated with truth membership, indeterminacy membership and falsity membership degrees. This classification helps in modeling uncertainty more effectively by capturing the nuances of indeterminate and inconsistent information within a triangular framework. The detailed definition and operations of TriNFN are described as follows.

Definition 8. [16] Let $N=(a,b,c;T,I,F)$ is single-valued TriNFN, consist of truth, indeterminacy and falsity membership function which defined as,

$$T(x)=\begin{cases}\dfrac{x-a}{b-a}T,\ a\le x\le b\\ T,\ x=b\\ \dfrac{c-x}{c-b}T,\ b<x\le c\\ 0,\ otherwise.\end{cases} \tag{10}$$

$$I(x)=\begin{cases}\dfrac{b-x+I(x-a)}{b-a},\ a\le x\le b\\ I,\ x=b\\ \dfrac{x-b+I(c-x)}{c-b},\ b<x\le c\\ 0,\ otherwise.\end{cases} \tag{11}$$

$$F(x)=\begin{cases}\dfrac{b-x+F(x-a)}{b-a},\ a\le x\le b\\ I,\ x=b\\ \dfrac{x-b+F(c-x)}{c-b},\ b<x\le c\\ 0,\ otherwise.\end{cases} \tag{12}$$

Definition 9. [17] Let $X_1=(a_1,b_1,c_1;T_1,I_1,F_1)$ and $X_2=(a_2,b_2,c_2;T_2,I_2,F_2)$ be TriNFN. The addition and multiplication operations are as below:

$$X_1+X_1=(a_1+a_2,b_1+b_2,c_1+c_2;T_1+T_2-T_1T_2,I_1I_2,F_1F_2), \tag{13}$$

$$X_1\times X_1=(a_1a_2,b_1b_2,c_1c_2;T_1T_2,I_1+I_2-I_1I_2,F_1+F_2-F_1F_2) \tag{14}$$

2.3 Kronecker Products and Vec-operator

Kronecker products and Vec-operator are widely used in solving matrix equations and in this study, these operators are used to convert the neutrosophic fuzzy matrix equations to a simpler form of neutrosophic fuzzy linear equations. Basically, the symbol of the Kronecker product is $\otimes_k$ and the definition is given as follows.

Definition 10. [18] Let matrix $A=[a_{ij}]$ is $m\times n$ and $B=[b_{ij}]$ is $p\times q$ then the Kronecker product of $A\otimes_k B$ is given by,

$$A_{m\times n}\otimes_k B_{p\times q}=\begin{pmatrix}a_{11}B & \cdots & a_{1n}B\\ \vdots & \ddots & \vdots\\ a_{m1}B & \cdots & a_{mn}B\end{pmatrix} \tag{15}$$

which is a $mp \times nq$ matrix.

2.4 Associated Linear System

Associated linear system (ALS) is developed to form a crisp form of the matrix equation. The definition of ALS is given as follows.

Definition 11. [22] Let $\tilde{S} = (m^{\tilde{S}}, \alpha^{\tilde{S}}, \beta^{\tilde{S}})$ and $\tilde{C} = (m^{\tilde{C}}, \alpha^{\tilde{C}}, \beta^{\tilde{C}})$ be positive, negative or near-zero fuzzy matrices, and $\tilde{X} = (m^{\tilde{X}}, \alpha^{\tilde{X}}, \beta^{\tilde{X}})$ be the solution which is in a positive fuzzy matrix. If $\tilde{S}$ is positive, then the forms of ALS obtained in matrix form, such that,

$$\begin{pmatrix} m^{\tilde{S}} & 0 & 0 \\ \alpha^{\tilde{S}} & (m^{\tilde{S}} - \alpha^{\tilde{S}}) & 0 \\ \beta^{\tilde{S}} & 0 & (m^{\tilde{S}} + \beta^{\tilde{S}}) \end{pmatrix} \begin{pmatrix} m^{\tilde{X}} \\ \alpha^{\tilde{X}} \\ \beta^{\tilde{X}} \end{pmatrix} = \begin{pmatrix} m^{\tilde{C}} \\ \alpha^{\tilde{C}} \\ \beta^{\tilde{C}} \end{pmatrix}. \tag{16}$$

2.5 Score Function Method

The common approach used to transform neutrosophic fuzzy number into crisp numerical values is the score function method. This method assigns a single representative value to each neutrosophic number, enabling straightforward comparison and ranking among multiple alternatives. The direct comparison between the components of neutrosophic fuzzy numbers, which are *T, I* and *F*, might be complex so the score function simplifies this process by integrating these components into one distinct and interpretable value [23].

Let $M = (T, I, F)$ be the singled-valued neutrosophic numbers. Then, the singled-valued neutrosophic score function is defined as,

$$s(M) = \frac{T + (1 - I) + (1 - F)}{3} = \frac{2 + T - I - F}{3}, \tag{17}$$

where $s : M \to [0,1]$.

2.6 Relative Residual Error

Relative residual error is the most common metric used in numerical linear algebra, which applicable to determine the accuracy of the solution for solving linear systems, $Ax = b$ [25]. By comparing the magnitude of residual to the magnitude of the original vector *b*, the relative residual error is calculated as

$$\text{Relative residual error} = \frac{\|Ax - b\|}{\|b\|}. \tag{18}$$

However, in this study of FFNSME the relative residual error measure how well the calculated solution $\tilde{X}$ satisfies the original Eq. [27].

Definition 12. [26] Given that $\tilde{X}$ is approximate solution of FFNSME, then the absolute residual, $\|R\|$ is given as follows,

$$\|R\| = \|\tilde{A}\tilde{X} + \tilde{X}\tilde{B} - \tilde{C}\|, \tag{19}$$

then, the relative residual, e is equal to

$$\begin{aligned} e &= \frac{\|R\|_F}{\|C\|_F} \\ &= \frac{\|\tilde{A}\tilde{X} + \tilde{X}\tilde{B} - \tilde{C}\|_F}{\|C\|_F} \end{aligned} \tag{20}$$

Here, $\|.\|_F$ denotes the Frobenius norm, which used to calculate the distance of each element from neutrosophic zero [26].

3. Methodology

The steps-by-steps algorithm is presented in this section.

Step 1: FFNSME is reduced to fully fuzzy neutrosophic linear system (FFNLS)

The FFNSME is reduced to FFNLS, $\tilde{D}\tilde{X} = \tilde{C}$ by using the fuzzy neutrosophic Kronecker product and neutrosophic fuzzy Vec-operator as defined in Theorem 1 below.

Theorem 1. Let $\tilde{A}$ and $\tilde{X}$ be the square matrices with the same order, such that $\tilde{A} = [\tilde{a}_{ij}]_{m\times m}$, $\tilde{X} = [\tilde{x}_{ij}]_{m\times m}$ and $\tilde{C} = [\tilde{c}_{ij}]_{m\times m}$, then FFNSME of $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ is equal to fully fuzzy neutrosophic linear systems

$$\tilde{D}\tilde{X} = \tilde{C}$$

where $\tilde{D} = \tilde{U}_{n\times n} \otimes \tilde{A}_{m\times m} + \tilde{B}_{m\times m} \otimes_k \tilde{U}_{n\times n}$, $\tilde{X} = Vec(\tilde{X})_{mn\times 1}$ and $\tilde{C} = Vec(\tilde{C})_{mn\times 1}$.

Previously, the multiplication operation of TriNFN is calculated based on the formula as in Eq. (10). In this study, since the LR-TriNFN is used as the coefficient of FFNSME which in the form of $(m,\alpha,\beta;T,I,F)$, then the multiplication operation of Eq. (7) is adapted into the existing operator of Eq. (14). Therefore, the new multiplication operator is defined as follows.

Definition 13. Let $X_1 = (m_1, \alpha_1, \beta_1; T_1, I_1, F_1)$ and $X_2 = (m_2, \alpha_2, \beta_2; T_2, I_2, F_2)$ be the positive LR-TriNFN. The multiplication arithmetic operator is defined as below.

$$X_1 \times X_2 = (m_1 m_2, m_1 m_2 - Min[(m_1 - \alpha_1)(m_2 - \alpha_2), (m_1 + \beta_1)(m_2 - \beta_2)], Max[(m_1 - \alpha_1)(m_1 + \beta_1), (m_1 + \beta_1)(m_2 - \beta_2)] - m_1 m_2; T_1 T_2, I_1 + I_2 - I_1 I_2, F_1 + F_2 - F_1 F_2). \tag{21}$$

Besides that, the additional operation of LR-TriNFN is executed according to the Eq. (13), such that

$$X_1 + X_2 = (m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2; T_1 + T_2 - T_1 T_2, I_1 I_2, F_1 F_2). \tag{22}$$

Next, the unitary fuzzy matrix is extended to include truth, indeterminacy, and falsity components as in the following theorem.

Theorem 2. Let $\tilde{U}$ be the fuzzy neutrosophic identity matrix which can be described as below,

$$\tilde{U} = \begin{pmatrix} (1,0,0;1,0,0) & (0,0,0;0,1,1) & \cdots & (0,0,0;0,1,1) \\ (0,0,0;0,1,1) & (1,0,0;1,0,0) & \cdots & (0,0,0;0,1,1) \\ \vdots & \vdots & \ddots & \vdots \\ (0,0,0;0,1,1) & (0,0,0;0,1,1) & \cdots & (1,0,0;1,0,0) \end{pmatrix}. \tag{23}$$

$\tilde{U}$ satisfies the condition of

(i) $\tilde{A}\tilde{U} = \tilde{U}\tilde{A} = \tilde{A}$

(ii) $\tilde{U}^T = \tilde{U}$

Step 2: FFNLS is deneutrosophicated to crisp form

i. Component of LR-TriFN of (m, α, β).

The conversion of the LR-TriFN is aimed to form an associated linear system as defined in Definition 11. Next, the following sub-matrices of $(m_{\tilde{D}} - \alpha_{\tilde{D}})$ and $(m_{\tilde{D}} + \beta_{\tilde{D}})$ are defined. All the sub-matrices are in the form of crisp matrices. Subsequently, all sub-matrices are substituted into a general form of an associated linear system as given in the Eq. (16). Thereby a crisp form of linear system

$$D_1 X_1 = C_1 \tag{24}$$

is established, which can be equivalent to

$$\begin{pmatrix} m^{\tilde{D}_1} & 0 & 0 \\ \alpha^{\tilde{D}_1} & (m^{\tilde{D}_1} - \alpha^{\tilde{D}_1}) & 0 \\ \beta^{\tilde{D}_1} & 0 & (m^{\tilde{D}_1} + \beta^{\tilde{D}_1}) \end{pmatrix} \begin{pmatrix} m^{\tilde{X}_1} \\ \alpha^{\tilde{X}_1} \\ \beta^{\tilde{X}_1} \end{pmatrix} = \begin{pmatrix} m^{\tilde{C}_1} \\ \alpha^{\tilde{C}_1} \\ \beta^{\tilde{C}_1} \end{pmatrix} \tag{25}$$

ii. Component of neutrosophic numbers (*T, I, F*).
Meanwhile, the conversion of the neutrosophic component (*T, I, F*) is executed by applying the score function method as stated in Eq. (17). Generally, the conversion of $(T_{11}^{\tilde{D}}, I_{11}^{\tilde{D}}, F_{11}^{\tilde{D}})$ to a crisp form is

$$s(T_{11}^{\tilde{D}}, I_{11}^{\tilde{D}}, F_{11}^{\tilde{D}}) = \frac{T_{11}^{\tilde{D}} + (1 - I_{11}^{\tilde{D}}) + (1 - F_{11}^{\tilde{D}})}{3} = D_{11}. \tag{26}$$

Hence, the following crisp of linear system is obtained,

$$\begin{pmatrix} D_{11} & \cdots & D_{1n} \\ \vdots & \ddots & \vdots \\ D_{m1} & \cdots & D_{mn} \end{pmatrix} \begin{pmatrix} x_{11} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{pmatrix} \tag{27}$$

which can be represented as

$$\mathrm{D}_2\mathrm{X}_2 = \mathrm{C}_2. \tag{28}$$

Step 3: Obtaining solution $\tilde{X}$

Following Step 2, there are two resulting solutions, $\tilde{\mathrm{X}}_1$ and $\tilde{\mathrm{X}}_2$. Then, both solutions are integrated to form the final solution, $\tilde{\mathrm{X}}$.

First, the solution $\tilde{X}_1$ is obtained by

$$\begin{pmatrix} m^{\tilde{X}_1} \\ \alpha^{\tilde{X}_1} \\ \beta^{\tilde{X}_1} \end{pmatrix} = \begin{pmatrix} m^{\tilde{D}_1} & 0 & 0 \\ \alpha^{\tilde{D}_1} & (m^{\tilde{D}_1} - \alpha^{\tilde{D}_1}) & 0 \\ \beta^{\tilde{D}_1} & 0 & (m^{\tilde{D}_1} + \beta^{\tilde{D}_1}) \end{pmatrix}^{-1} \begin{pmatrix} m^{\tilde{C}_1} \\ \alpha^{\tilde{C}_1} \\ \beta^{\tilde{C}_1} \end{pmatrix}. \tag{29}$$

Hence, the solution can be written as

$$\tilde{X}_1 = \begin{pmatrix} (m_{11}^{X_1}, \alpha_{11}^{X_1}, \beta_{11}^{X_1}) & (m_{12}^{X_1}, \alpha_{12}^{X_1}, \beta_{12}^{X_1}) \\ (m_{21}^{X_1}, \alpha_{21}^{X_1}, \beta_{21}^{X_1}) & (m_{22}^{X_1}, \alpha_{22}^{X_1}, \beta_{22}^{X_1}) \end{pmatrix} \tag{30}$$

Meanwhile, the solution X_2 will be in the form of crisp matrix which obtain as follows,

$$\begin{pmatrix} x_{11} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} D_{11} & \dots & D_{1n} \\ \vdots & \ddots & \vdots \\ D_{m1} & \cdots & D_{mn} \end{pmatrix}^{-1} \begin{pmatrix} c_{11} \\ \vdots \\ c_{nn} \end{pmatrix} \quad (31)$$

Hence, the conversion of crisp matrix of X_2 to neutrosophic matrix is required. However, to the best of our knowledge, there is currently no literature that provides a formal approach for converting a crisp number into a neutrosophic representation. Therefore, for the sake of simplicity and illustrative purposes, this study considers the obtained value of X_2 as the truth-membership value (T) while the indeterminacy (I) and falsity (F) components are assigned based on reasonable assumptions or approximations. It is noted that, the assumption and approximations value must ensure that the basic fundamental of neutrosophic number is imposed, which allows numbers within $[0,1]$, with $0+ \leq T + I + F \leq 3+$ as stated earlier in the Equation (9).

Therefore, the solution $\tilde{X}_2$ after represented in a neutrosophic form, can be generally written as,

$$\tilde{X}_2 = \begin{pmatrix} (T_{11}^{\tilde{X}_2}, I_{11}^{\tilde{X}_2}, F_{11}^{\tilde{X}_2}) & \dots & (T_{1n}^{\tilde{X}_2}, I_{1n}^{\tilde{X}_2}, F_{1n}^{\tilde{X}_2}) \\ \vdots & \ddots & \vdots \\ (T_{m1}^{\tilde{X}_2}, I_{m1}^{\tilde{X}_2}, F_{m1}^{\tilde{X}_2}) & \cdots & (T_{mn}^{\tilde{X}_2}, I_{mn}^{\tilde{X}_2}, F_{mn}^{\tilde{X}_2}) \end{pmatrix}. \quad (32)$$

4. Results

4.1 Numerical Example

Example 1: Consider the FFNSME, $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$, where

$$\tilde{A} = \begin{pmatrix} (11,4,3;0.3,0.5,0.1) & (5,1,3;0.9,0.7,0.2) \\ (7,2,1;0.8,0.2,0.4) & (8,6,9;0.6,0.4,0.3) \end{pmatrix}, \tilde{B} = \begin{pmatrix} (8,3,1;0.2,0.3,0.1) & (8,2,4;0.1,0.2,0.2) \\ (8,6,5;0.4,0.1,0.3) & (5,1,2;0.5,0.3,0.2) \end{pmatrix}$$

$$\tilde{C} = \begin{pmatrix} (134,82,286;0.7,0.13,0.2) & (221,157,443;0.83,0.13,0.22) \\ (123,65,322;0.87,0.14,0.08) & (191,109,453;0.89,0.18,0.09) \end{pmatrix}$$

and the fully fuzzy neutrosophic solution, $\tilde{X}$ is

$$\tilde{X} = \begin{pmatrix} (m_{11}^X, \alpha_{11}^X, \beta_{11}^X; T_{11}^X, I_{11}^X, F_{11}^X) & (m_{12}^X, \alpha_{12}^X, \beta_{12}^X; T_{12}^X, I_{12}^X, F_{12}^X) \\ (m_{21}^X, \alpha_{21}^X, \beta_{21}^X; T_{21}^X, I_{21}^X, F_{21}^X) & (m_{22}^X, \alpha_{22}^X, \beta_{22}^X; T_{22}^X, I_{22}^X, F_{22}^X) \end{pmatrix}$$

Step 1. Reduce the FFNSME of $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ into FFNLS, $\tilde{D}\tilde{X} = \tilde{C}$. First, $\tilde{U}_{2\times2} \otimes_k \tilde{A}_{2\times2}$.

$$\left(\tilde{U}_{(2\times2)} \otimes_k \tilde{A}_{(2\times2)}\right) = \begin{pmatrix} (11,4,3;0.3,0.5,0.1) & (5,1,3;0.9,0.7,0.2) & (0,0,0;0,1,1) & (0,0,0;0,1,1) \\ (7,2,1;0.8,0.2,0.4) & (8,6,9;0.6,0.4,0.3) & (0,0,0;0,1,1) & (0,0,0;0,1,1) \\ (0,0,0;0,1,1) & (0,0,0;0,1,1) & (11,4,3;0.3,0.5,0.1) & (5,1,3;0.9,0.7,0.2) \\ (0,0,0;0,1,1) & (0,0,0;0,1,1) & (7,2,1;0.8,0.2,0.4) & (8,6,9;0.6,0.4,0.3) \end{pmatrix}$$

Similarly, for $\tilde{B}^T_{2\times2} \otimes_k \tilde{U}_{2\times2}$,

$$\left(\tilde{B}^T_{(2\times2)} \otimes_k \tilde{U}_{(2\times2)}\right) = \begin{pmatrix} (8,3,1;0.2,0.3,0.1) & (0,0,0;0,1,1) & (8,6,5;0.4,0.1,0.3) & (0,0,0;0,1,1) \\ (0,0,0;0,1,1) & (8,3,1;0.2,0.3,0.1) & (0,0,0;0,1,1) & (8,6,5;0.4,0.1,0.3) \\ (8,2,4;0.1,0.2,0.2) & (0,0,0;0,1,1) & (11,4,3;0.3,0.5,0.1) & (0,0,0;0,1,1) \\ (0,0,0;0,1,1) & (8,2,4;0.1,0.2,0.2) & (0,0,0;0,1,1) & (8,6,9;0.6,0.4,0.3) \end{pmatrix}$$

Next, execute $\tilde{U}_{2\times2} \otimes_k \tilde{A}_{2\times2} \oplus \tilde{B}^T_{2\times2} \otimes_k \tilde{U}_{2\times2}$ to obtain $\tilde{D}$ and FFNLS, $\tilde{D}\tilde{X} = \tilde{C}$ is formed as follows,

$$\begin{pmatrix} (19,7,4;0.44,0.15,0.01) & (5,1,3;0.9,0.7,0.2) & (8,6,5;0.4,0.1,0.3) & (0,0,0;0,1,1) \\ (7,2,1;0.8,0.2,0.4) & (16,9,10;0.68,0.12,0.03) & (0,0,0;0,1,1) & (8,6,5;0.4,0.1,0.3) \\ (8,2,4;0.1,0.2,0.2) & (0,0,0;0,1,1) & (16,5,5,0.65,0.15,0.02) & (5,1,3;0.9,0.7,0.2) \\ (0,0,0;0,1,1) & (8,2,4;0.1,0.2,0.2) & (7,2,1;0.8,0.2,0.4) & (13,7,11;0.8,0.12,0.06) \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{pmatrix}$$
$$= \begin{pmatrix} (134,82,286;0.7,0.13,0.2) \\ (221,157,443;0.87,0.14,0.08) \\ (123,65,322;0.83,0.13,0.22) \\ (191,109,453;0.89,0.18,0.09) \end{pmatrix}$$

Step 2. The deneutrosophication of the FFNLS, $\tilde{D}\tilde{X} = \tilde{C}$ is applied by decomposing the FFNLS into crisp subsystems corresponding to LR-TriFN (m,α,β). The extracted crisp coefficient matrices are as follows,

$$m^{\tilde{D}} = \begin{pmatrix} 19 & 5 & 8 & 0 \\ 7 & 16 & 0 & 8 \\ 8 & 0 & 16 & 5 \\ 0 & 6 & 7 & 13 \end{pmatrix}, \quad \alpha^{\tilde{D}} = \begin{pmatrix} 7 & 1 & 6 & 0 \\ 2 & 9 & 0 & 6 \\ 2 & 0 & 5 & 1 \\ 0 & 2 & 2 & 7 \end{pmatrix}, \quad \beta^{\tilde{D}} = \begin{pmatrix} 4 & 3 & 5 & 0 \\ 1 & 10 & 0 & 5 \\ 4 & 0 & 5 & 3 \\ 0 & 4 & 1 & 11 \end{pmatrix}$$

$$m^{\tilde{c}} = \begin{pmatrix} 134 \\ 221 \\ 123 \\ 191 \end{pmatrix}, \quad \alpha^{\tilde{c}} = \begin{pmatrix} 82 \\ 157 \\ 65 \\ 109 \end{pmatrix}, \quad \beta^{\tilde{c}} = \begin{pmatrix} 286 \\ 443 \\ 322 \\ 453 \end{pmatrix}.$$

From there, the crisp form of matrix of $D_1X_1 = C_1$ which based on associated linear system form of Eq. (29) is obtained.

On the other hand, the conversion of the neutrosophic number in the $DX = C$, is implemented using the score function formula of Eq. (17). Here, the neutrosophic number of $\left(T_{11}, I_{11}, F_{11}\right) = (0.51, 0.25, 0.01)$ is illustrated.

$$s\left(T_{11}, I_{11}, F_{11}\right) = \frac{T_{11} + (1 + I_{11}) + (1 - F_{11})}{3}$$
$$= \frac{0.44 + (1 - 0.15) + (1 - 0.091)}{3}$$
$$= 0.76$$

The same process is executed for all the elements of the matrix D and X. Hence, the crisp linear system in the form of $D_2X_2 = C_2$ is obtained as follows,

$$\begin{pmatrix} 0.76 & 0.67 & 0.67 & 0 \\ 0.73 & 0.84 & 0 & 0.67 \\ 0.57 & 0 & 0.83 & 0.67 \\ 0 & 0.57 & 0.73 & 0.87 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0.79 \\ 0.88 \\ 0.83 \\ 0.87 \end{pmatrix} \tag{33}$$

Step 3. According to the crisp form of $D_1X_1 = C_1$ the solution of X_1 is determined by taking the inverse coefficient of D_1, which can be expressed in the following term,

$$\tilde{X}_1 = \begin{pmatrix} (m_{11}^{X_1}, \alpha_{11}^{X_1}, \beta_{11}^{X_1}) & (m_{12}^{X_1}, \alpha_{12}^{X_1}, \beta_{12}^{X_1}) \\ (m_{21}^{X_1}, \alpha_{21}^{X_1}, \beta_{21}^{X_1}) & (m_{22}^{X_1}, \alpha_{22}^{X_1}, \beta_{22}^{X_1}) \end{pmatrix} = \begin{pmatrix} (3,1,2) & (4,2,9) \\ (9,3,8) & (7,1,7) \end{pmatrix} \tag{34}$$

Furthermore, the solution of X_2 in Eq. (33) is expressed into the neutrosophic form (*T, I, F*) by directly assigning the value obtained as the component *T*. While for the other component, *I* and *F* the values are determined by assuming or approximating, which ensures that the resulting satisfy the fundamental condition of neutrosophic.

$$\tilde{X}_2 = \begin{pmatrix} (T_{11}^{X_2}, I_{11}^{X_2}, F_{11}^{X_2}) & (T_{12}^{X_2}, I_{12}^{X_2}, F_{12}^{X_2}) \\ (T_{21}^{X_2}, I_{21}^{X_2}, F_{21}^{X_2}) & (T_{22}^{X_2}, I_{22}^{X_2}, F_{22}^{X_2}) \end{pmatrix} = \begin{pmatrix} (0.38,0.67,0.58) & (0.37,0.72,0.54) \\ (0.3,0.66,0.59) & (0.44,0.72,0.4) \end{pmatrix} \tag{35}$$

Finally, both fuzzy and neutrosophic elements from Eq. (34) and (35), respectively, are gathered into a single matrix to form a triangular neutrosophic fuzzy solution, $\tilde{X}$ as the following,

$$\tilde{X} = \begin{pmatrix} (3,1,2;0.38,0.67,0.58) & (4,2,9;0.37,0.72,0.54) \\ (9,3,8;0.3,0.66,0.59) & (7,1,7;0.44,0.72,0.4) \end{pmatrix} \tag{36}$$

4.1 Verification of the Solution

The verification of the obtained solution $\tilde{X}$ is based on the numerical results presented in the previous section. The accuracy of the results is validated by substituting the computed solution back into the FFNSME of the Example 1, as follows.

$$\tilde{A}\tilde{X}=\begin{pmatrix}(11,4,3;0.3,0.5,0.1) & (5,1,3;0.9,0.7,0.2)\\(7,2,1;0.8,0.2,0.4) & (8,6,9;0.6,0.4,0.3)\end{pmatrix}\begin{pmatrix}(3,1,2;0.38,0.67,0.58) & (4,2,9;0.37,0.72,0.54)\\(9,3,8;0.3,0.66,0.59) & (7,1,7;0.44,0.72,0.4)\end{pmatrix}$$
$$=\begin{pmatrix}(78,40,128;0.438,0.709,0.588) & (79,41,,215;0.497,0.688,0.361)\\(93,71,236;0.424,0.517,0.655) & (84,62,258;0.529,0.484,0.489)\end{pmatrix}$$
$$\tilde{X}\tilde{B}=\begin{pmatrix}(3,1,2;0.38,0.67,0.58) & (4,2,9;0.37,0.72,0.54)\\(9,3,8;0.3,0.66,0.59) & (7,1,7;0.44,0.72,0.4)\end{pmatrix}\begin{pmatrix}(8,3,1;0.2,0.3,0.1) & (8,2,4;0.1,0.2,0.2)\\(8,6,5;0.4,0.1,0.3) & (5,1,2;0.5,0.3,0.2)\end{pmatrix}$$
$$=\begin{pmatrix}(56,42,158;0.417,0.444,0.532) & (44,24,107;0.438,0.585,0.509)\\(128,86,207;0.483,0.292,0.503) & (107,47,195;0.497,0.457,0.490)\end{pmatrix}$$

Then, $\tilde{A}\tilde{X}+\tilde{X}\tilde{B}=\tilde{C}*$ is equal to,

$$\tilde{C}*=\begin{pmatrix}(134,82,286;0.67,0.31,0.3) & (123,65,322;0.71,0.4,0.18)\\(221,157,443;0.7,0.15,0.33) & (191,109,453;0.76,0.22,0.23)\end{pmatrix}$$

According to the above, the result of $\tilde{C}*$ yields a matrix that is approximately equal to the original matrix $\tilde{C}$, as shown in the Example 1. To quantify the precision of the solution, the relative residual error as mentioned in Definition 12 is applied. In this study, the relative residual error is calculated as follows,

Relative residual error, $e=\dfrac{\left\|\tilde{C}*-\tilde{C}\right\|_F}{\left\|\tilde{C}\right\|_F}$.

First, the crisp form of $\tilde{C}*$ and $\tilde{C}$ are obtained as follows,

$$\tilde{C}*=\begin{pmatrix}0.69 & 0.71\\0.74 & 0.77\end{pmatrix}$$
$$\tilde{C}=\begin{pmatrix}0.79 & 0.88\\0.83 & 0.87\end{pmatrix} \tag{37}$$

Based on the calculation, norm which is the single value of the matrix in Eq. (37) is calculated.

$$\left\|\tilde{C}^{*}-\tilde{C}\right\|_{F}=\sqrt{(0.13)^{2}+(0.19)^{2}+(0.11)^{2}+(0.11)^{2}}$$
$$=0.278$$
$$\left\|\tilde{C}\right\|_{F}=\sqrt{(0.79)^{2}+(0.88)^{2}+(0.83)^{2}+(0.87)^{2}}$$
$$=1.687$$

then,

$$e=\frac{0.278}{1.687}$$
$$=0.16$$

Although the tolerance value of $e=0.16$ appears relatively high, it should be considered due to the preliminary approximations made for the component *I* and *F* as shown in the Eq. (35). Currently, no formal literature provides a standard for converting crisp results back into neutrosophic form, requiring assumptions that naturally influence the final residual. Furthermore, these deviations are attributed to the algebraic complexities of neutrosophic multiplication and inherent floating-point round-off errors. Despite this, the proposed algorithm is still can be acceptable and applicable, offering a flexible representation of uncertainty that extends the capabilities of classical or fuzzy Sylvester approaches.

5. Conclusion

In conclusion, this study has developed a framework to solve FFNSME, which generalizes the classical Sylvester formulation to incorporate truth, indeterminacy and falsity components. By applying neutrosophic representations with correspondent solution methods, the new method is capable to accommodate uncertainties as well as capturing inconsistencies that cannot be presented using conventional or fuzzy models. The accuracy of the solution was verified through relative residual error analysis. While an error of 0.16 was noted, it is considered acceptable and primarily attributed to floating-point arithmetic round-off and the complexities of neutrosophic multiplication. Ultimately, the neutrosophic Sylvester formulation provides a comprehensive and flexible tools for analysing the dynamic systems that deal with uncertainties. Future research should explore trapezoidal and pentagonal neutrosophic numbers and extend the framework to large-scale systems, focusing on the development of efficient numerical algorithms and the integration of optimization-based or iterative methods to enhance computational performance. Furthermore, to improve the current error margin of 0.16, subsequent studies will focus on developing a formal procedure for converting crisp values into neutrosophic representations and implementing high-precision iterative algorithms. These advancements are expected to enhance computational performance and reliability, particularly when scaling the framework to accommodate large-scale, high-dimensional, and time-varying dynamic systems.

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