SEMARAK ILMU PUBLISHING
202103268166(003316878-P)

Semarak International Journal of Machine Learning

Journal homepage:
https://semarakilmu.my/index.php/sijml/index
ISSN: 3030-5241

Fuzzy Predator-Prey Systems by Extended Runge-Kutta Method with Polynomial Interpolation Technique

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ARTICLE INFO

Article history:
Received 30 January 2024
Received in revised form 18 March 2024
Accepted 24 March 2024
Available online 26 April 2024

Keywords:
Predator-prey systems, fuzzy differential equation, extended Runge-Kutta method, polynomial interpolation

ABSTRACT

The growing significance of uncertainty quantification in mathematical modeling of physical phenomena is evident, with fuzzy sets emerging as an alternative approach. This paper focuses on exploring a numerical method for addressing fuzzy differential equations in two-dimensional problems like fuzzy predator-prey systems. The numerical solution employed the extended Runge-Kutta fourth-order method. To overcome challenges related to polynomial fuzzy differential equations, a combination of the polynomial interpolation technique and the extended Runge-Kutta fourth-order method was applied, mitigating the issues associated with high-degree polynomials. The proposed approach for fuzzy predator-prey systems is briefly described, accompanied by a numerical example. The obtained results underscore the efficacy of the extended Runge-Kutta fourth-order method with polynomial interpolation, showcasing its remarkable accuracy and potential as an alternative method for addressing various uncertainty-related problems.

1. Introduction

Differential equations are widely used in applied science and engineering to represent real life problems and systems. In many practical cases, uncertainties can appear which need to take into account to improve the accuracy and predictability of estimates. Since Zadeh [1] introduced the concept of fuzzy set and corresponding fuzzy operations, enormous efforts have been made to develop various aspects of the theory and applications of fuzzy systems.

The study of this topic has increased in recent years. A new fuzzy inference for diabetes classification was proposed by Lixandru-Petre, [2]. It was followed up by Gupta, *et al*., [3] describes the fuzzy logic-based systems for the diagnosis of various diseases. In their article also focuses on various fuzzy models that are being used in healthcare systems for making decisions. Research continues, both in terms of the theory of fuzzy and its application to real-world problems by several authors [4-8].

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https://doi.org/10.37934/sijml.1.1.1119a

In this paper, the predator-prey model is proposed in 2-dimensional FDEs. Since the predator-prey model in the fuzzy setup provides a more realistic depiction of the phenomena, the initial conditions are considered as fuzzy because the initial population estimates in a real situation may not be known accurately.

There are some authors [8-12] who have studied predator-prey models in which the initial populations of predator and prey are uncertainty. They all gave numerical solutions to differential equations with fuzzy initial conditions and some discussed the stability of the solutions. Ahmad & De Baets [9] applied a Runge-Kutta fourth order method through the principles of Zadeh extension for some numerical simulations of the predator-prey population model. Sukarsih *et al.,* [13] also used a Runge-Kutta method for solving two predator-prey models with a Holling type II functional.

In order to solve the fuzzy predator-prey system (FPPS), some limitations need to be addressed. From the basic operation of fuzzy with polynomial operation, degree of polynomial will develop very fast. In this paper, the extended Runge-Kutta fourth order (ERK4) method has been incorporated with polynomial interpolation technique to reduce the degree of polynomial. Semi exact solution with crisp solution has been employed for comparison with numerical solution.

The paper is organized in the following manner in the next section model description are presented, in the section 3 we describe the numerical solution of the model. Illustrative example is given at the section 4. The last section will contain some conclusions.

2. Model Description

The population model between predators and preys can be represented as a system of two coupled first order nonlinear differential equations which is also known as Lotka-Volterra equation,

$$\begin{aligned} x'(t) &= ax - bxy; \\ y'(t) &= -cx + dxy; \end{aligned} . \tag{1}$$

with the initial condition $x(t_0) = x_0$ and $y(t_0) = y_0$ where a, b, c, and d are positive real parameters describing the interaction of the two species, $x(t)$ denotes the population of the prey species and $y(t)$ denotes the population of the predator species, x_0 and y_0 are the initial estimates of the species.

For the system as given by Eq. (1), it may not be possible to obtain exact estimates of initial population, thus the initial estimates are represented by fuzzy numbers. By considering Eq. (1) is a predator-prey system with fuzzy initial condition $x(t_0) = x_0$ and $y(t_0) = y_0$ where $x'(t)$ and $y'(t)$ are continuous functions. The α-cuts of $x(t)$ and $y(t)$ are:

$$\begin{aligned} x_i(t,\alpha) &= \left[\underline{x_i}(t,\alpha), \overline{x_i}(t,\alpha)\right], \\ y_i(t,\alpha) &= \left[\underline{y_i}(t,\alpha), \overline{y_i}(t,\alpha)\right], \end{aligned} . \tag{2}$$

for $i = 1, 2, \ldots, n$ and $\alpha \in [0, 1]$.

3. Numerical Solution of the Model

The general form of an explicit Runge-Kutta method is expressed as Lambert [14],

$$
\begin{aligned}
y_{n+1} &= y_n + h\sum_{j=1}^{m} b_j k_j, \\
k_j &= f\left(y_n + h\sum_{s=1}^{j-1} a_{js} k_s \right),\ j = 1, 2, \cdots, m.
\end{aligned}
\quad . \tag{3}
$$

Most efforts to improve the order of Runge–Kutta methods have been made by increasing number of Taylor's series expansion. This increases the number of function evaluations accordingly as highlighted by some researches [15-19]. Goeken and Johnson [20] and Wu and Xia [21] using higher derivatives to proposed a class of Runge–Kutta methods, and new third, fourth and fifth order numerical methods were presented. Specifically, f' is embedded in f i.e. f' is approximated by a difference quotient of past and current evaluations of , $f(y_n) \approx \frac{(f(y_n) - f(y_{n-1}))}{h}$.

This motivates a family of extended Runge–Kutta formula by Wu & Xia [21] in the form of

$$
\begin{aligned}
y_{n+1} &= f\left(y_n + \sum_{j=1}^{m} b_j k_j^{(1)} + h^2 \sum_{j=1}^{m} c_j k_j^{(2)} \right), \\
k_j^{(1)} &= f\left(y_n + h\sum_{s=1}^{j-1} a_{js} k_s^{(1)} \right), \\
k_j^{(2)} &= f'\left(y_n + h\sum_{s=1}^{j-1} b_{js} k_s^{(1)} \right),\ j = 1, 2, \cdots, m.
\end{aligned}
\quad . \tag{4}
$$

3.1 Extended Runge-Kutta Fourth Order Method for Fuzzy Predator-Prey System

From Eq. (4), the ERK4 method for FPPS is defined

$$
\begin{aligned}
\underline{x}_{i+1} &= \underline{x}_i + hk_1^{(1)} + h^2\left(\frac{1}{6}\underline{k}_1^{(2)} + \frac{1}{3}\underline{k}_3^{(2)} \right), \\
\overline{x}_{i+1} &= \overline{x}_i + h\overline{k}_1^{(1)} + h^2\left(\frac{1}{6}\overline{k}_1^{(2)} + \frac{1}{3}\overline{k}_3^{(2)} \right), \\
y_{i+1} &= \underline{y}_i + h\underline{l}_1^{(1)} + h^2\left(\frac{1}{6}\underline{l}_1^{(2)} + \frac{1}{3}\underline{l}_3^{(2)} \right), \\
\overline{y}_{i+1} &= \overline{y}_i + h\overline{l}_1^{(1)} + h^2\left(\frac{1}{6}\overline{l}_1^{(2)} + \frac{1}{3}\overline{l}_3^{(2)} \right),
\end{aligned}
\tag{5}
$$

where

$$\underline{k}_1^{(1)}(x(t;\alpha)) = \min\{f(x(t;\alpha))\},$$
$$\overline{k}_1^{(1)}(x(t;\alpha)) = \max\{f(x(t;\alpha))\},$$
$$\underline{k}_2^{(1)}(x(t;\alpha)) = \min\left\{f\left(x(t;\alpha) + \frac{1}{4}h\underline{k}_1^{(1)}x(t;\alpha)\right)\right\},$$
$$\overline{k}_2^{(1)}(x(t;\alpha)) = \max\left\{f\left(x(t;\alpha) + \frac{1}{4}h\overline{k}_1^{(1)}x(t;\alpha)\right)\right\},$$
$$\underline{k}_3^{(1)}(x(t;\alpha)) = \min\{f(x(t;\alpha))\},$$
$$\overline{k}_3^{(1)}(x(t;\alpha)) = \max\{f(x(t;\alpha))\},$$
$$\underline{k}_1^{(2)}(x(t;\alpha)) = \min\{f'(x(t;\alpha))\},$$
$$\overline{k}_1^{(2)}(x(t;\alpha)) = \max\{f'(x(t;\alpha))\},$$
$$\underline{k}_3^{(2)}(x(t;\alpha)) = \min\left\{f'\left(x(t;\alpha) + \frac{1}{2}h\underline{k}_2^{(1)}x(t;\alpha)\right)\right\},$$
$$\overline{k}_3^{(2)}(x(t;\alpha)) = \max\left\{f'\left(x(t;\alpha) + \frac{1}{2}h\overline{k}_2^{(1)}x(t;\alpha)\right)\right\},$$

and

$$\underline{l}_1^{(1)}(x(t;\alpha)) = \min\{g(y(t;\alpha))\},$$
$$\overline{l}_1^{(1)}(x(t;\alpha)) = \max\{g(y(t;\alpha))\},$$
$$\underline{l}_2^{(1)}(x(t;\alpha)) = \min\left\{g\left(y(t;\alpha) + \frac{1}{4}h\underline{l}_1^{(1)}y(t;\alpha)\right)\right\},$$
$$\overline{l}_2^{(1)}(y(t;\alpha)) = \max\left\{g\left(y(t;\alpha) + \frac{1}{4}h\overline{l}_1^{(1)}y(t;\alpha)\right)\right\},$$
$$\underline{l}_3^{(1)}(y(t;\alpha)) = \min\{g(y(t;\alpha))\},$$
$$\overline{l}_3^{(1)}(y(t;\alpha)) = \max\{g(y(t;\alpha))\},$$
$$\underline{l}_1^{(2)}(y(t;\alpha)) = \min\{g'(y(t;\alpha))\},$$
$$\overline{l}_1^{(2)}(y(t;\alpha)) = \max\{g'(y(t;\alpha))\},$$
$$\underline{l}_3^{(2)}(y(t;\alpha)) = \min\left\{g'\left(y(t;\alpha) + \frac{1}{2}h\underline{l}_2^{(1)}y(t;\alpha)\right)\right\},$$
$$\overline{l}_3^{(2)}(y(t;\alpha)) = \max\left\{g'\left(y(t;\alpha) + \frac{1}{2}h\overline{l}_2^{(1)}y(t;\alpha)\right)\right\}.$$

Accordingly, the following is defined:

$$\begin{aligned}
F(x(t;\alpha)) &= h\underline{k}_1^{(1)}(x(t;\alpha)) + \frac{1}{6}h^2\underline{k}_1^{(2)}(x(t;\alpha)) + \frac{1}{3}h^2\underline{k}_3^{(2)}(x(t;\alpha)),\\
H(x(t;\alpha)) &= h\overline{k}_1^{(1)}(x(t;\alpha)) + \frac{1}{6}h^2\overline{k}_1^{(2)}(x(t;\alpha)) + \frac{1}{3}h^2\overline{k}_3^{(2)}(x(t;\alpha)),\\
I(y(t;\alpha)) &= h\underline{l}_1^{(1)}(y(t;\alpha)) + \frac{1}{6}h^2\underline{l}_1^{(2)}(y(t;\alpha)) + \frac{1}{3}h^2\underline{l}_3^{(2)}(y(t;\alpha)),\\
J(y(t;\alpha)) &= h\overline{l}_1^{(1)}(y(t;\alpha)) + \frac{1}{6}h^2\overline{l}_1^{(2)}(y(t;\alpha)) + \frac{1}{3}h^2\overline{l}_3^{(2)}(y(t;\alpha)).
\end{aligned} \tag{6}$$

Then, Eq. (6) is rewritten as:

$$\begin{aligned}
hF(x(t;\alpha)) &= \underline{k}_1^{(1)}(x(t;\alpha)) + \frac{1}{6}h\underline{k}_1^{(2)}(x(t;\alpha)) + \frac{1}{3}h\underline{k}_3^{(2)}(x(t;\alpha)),\\
hH(x(t;\alpha)) &= \overline{k}_1^{(1)}(x(t;\alpha)) + \frac{1}{6}h\overline{k}_1^{(2)}(x(t;\alpha)) + \frac{1}{3}h\overline{k}_3^{(2)}(x(t;\alpha)),\\
hI(y(t;\alpha)) &= \underline{l}_1^{(1)}(y(t;\alpha)) + \frac{1}{6}h\underline{l}_1^{(2)}(y(t;\alpha)) + \frac{1}{3}h\underline{l}_3^{(2)}(y(t;\alpha)),\\
hJ(y(t;\alpha)) &= \overline{l}_1^{(1)}(y(t;\alpha)) + \frac{1}{6}h\overline{l}_1^{(2)}(y(t;\alpha)) + \frac{1}{3}h\overline{l}_3^{(2)}(y(t;\alpha)).
\end{aligned} \tag{7}$$

3.2 Fuzzy Polynomial Interpolation

A fuzzy polynomial of degree *m* is a function of the form

$$P_m(\alpha) = a_0 + a_1\alpha + \cdots + a_m\alpha^m = y_m,\ m > n. \tag{8}$$

where $P_m(\alpha) = [\underline{P}_m(\alpha), \overline{P}_m(\alpha)], a_0, \cdots, a_m \in \mathbb{R}$ with m, n are positive integer. The higher order polynomial for $P_m(\alpha)$ will be regenerated into low order polynomial, $P_n(\alpha)$. In this paper, the value of n is set as 10. So that, $P_m(\alpha)$ will be regenerated to the 11 points data, $\alpha = [\ \alpha_0,\ \alpha_1,\ \alpha_2,\ \cdots, \alpha_{10}]$. Therefore, the data to be fitted are $[y_0, y_1, \cdots, y_{10}]$.

Zulkefli proved the following, which appears as Theorem 1 and Theorem 2 of [22].

Theorem 1 (Zulkefli, [22])
Let $P^m, m > n$, is a monotonic polynomial with degree m. Then P_n which is an interpolated polynomial with degree n, is a pointwise monotonic polynomial with degree n.

Theorem 2 (Zulkefli, [22])
The degenerate polynomial, P_n is unique.

Figure 1 is the schematic representation of the numerical solution for FPPS in a flowchart. The flowchart describes the ERK4 method incorporated with polynomial interpolation technique used for solving FPPS.

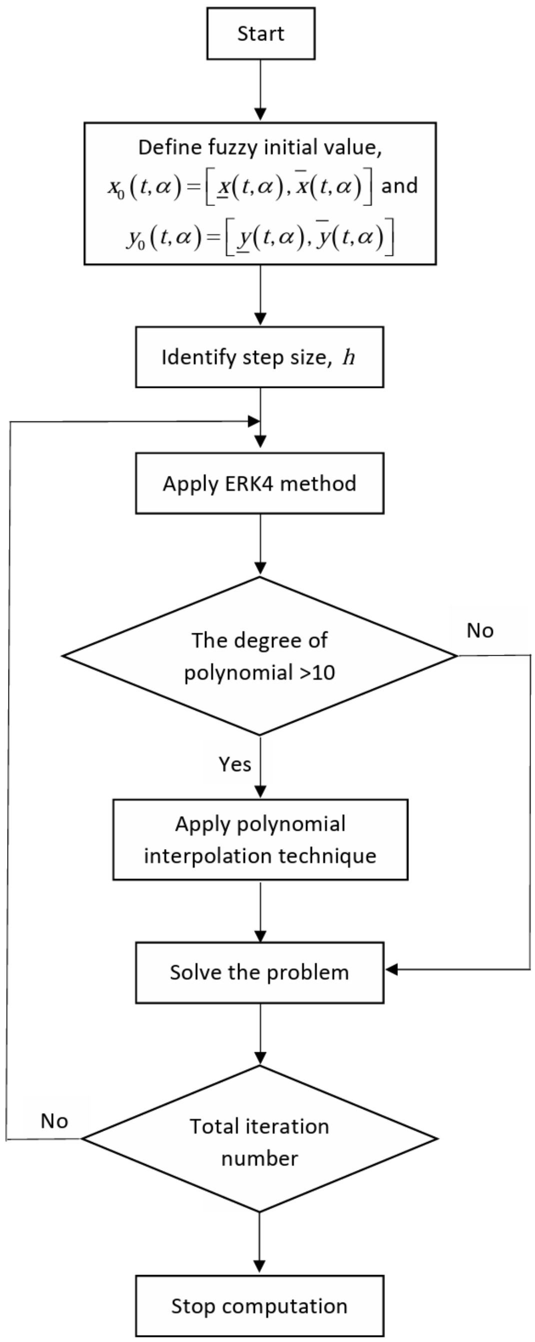

Fig. 1. Flowchart of numerical solution for FPPS

Firstly, fuzzy initial values $x_0(t,\alpha)=\left[\underline{x}(t,\alpha),\bar{x}(t,\alpha)\right]$ and $y_0(t,\alpha)=\left[\underline{y}(t,\alpha),\bar{y}(t,\alpha)\right]$ are assigned and the step size h is identified. Then, $x(t+h,\alpha)$ and $y(t+h,\alpha)$ are calculated by ERK4 method. During the calculation, if degree of polynomial is greater than 10, polynomial interpolation technique is applied in order to reduce the high degree polynomial to order 10. The stopping criteria is used when total iteration number is reached and the computation is ended. If the total iteration number is not reached, then the whole program will repeat start from statement "*apply ERK4 method*".

4. Results

In this section, the problem of FPPS taken from Ömer, *et.al* [10] was used as a test equation for the numerical approach developed in this paper. For this case, consider

$$\begin{aligned} x'(t) &= 0.1x - 0.005xy, \\ y'(t) &= -0.4y + 0.008xy, \\ x(0) &= [100 + 30\alpha, 160 - 30\alpha], \\ y(0) &= [20 + 20\alpha, 60 - 20\alpha]. \end{aligned} \tag{9}$$

where $x(t)$ and $y(t)$ are the numbers of prey and predator at time t.

The problems in Eq. (9) are complex to have the exact solution. Therefore, semi exact solution with crisp solution has been employed for comparison with the numerical solution. The semi exact solution can be obtained by linearization using Taylor's expansion as follows

$$\begin{aligned} X'(t) &= 0.1x_0 - 0.005x_0y_0 + (0.1 - 0.005y_0)X - 0.005x_0Y, \\ Y'(t) &= -0.4y_0 + 0.008x_0y_0 + 0.008y_0X + (-0.4 + 0.008x_0)Y, \\ x(0) &= x_0, y(0) = y_0, \\ X &= x - x_0, Y = y - y_0. \end{aligned}$$
(10)

Eq. (10) valid for small increment of X, Y with continuous of x_0, y_0 where the initial value is given $(x_0, y_0) = (130, 40)$. Eq. (10) is solved analytically. Numerical coding was performed in Microsoft Visual C++. By using ERK4 method in Eq. (5) with $h = 0.1$, the results for ERK4 method with polynomial interpolation technique and semi exact solution are illustrated in Figure 2 and Figure 3, respectively.

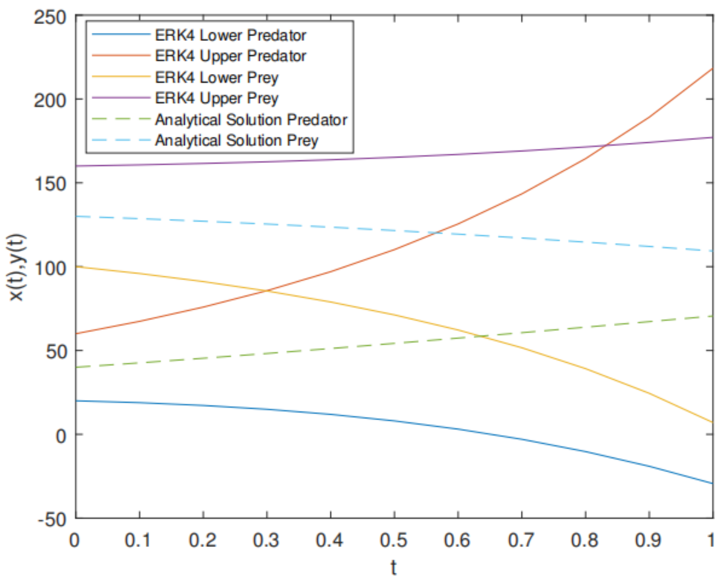

Fig. 2. Numerical result of interpolation ERK4 in 2D FPPS

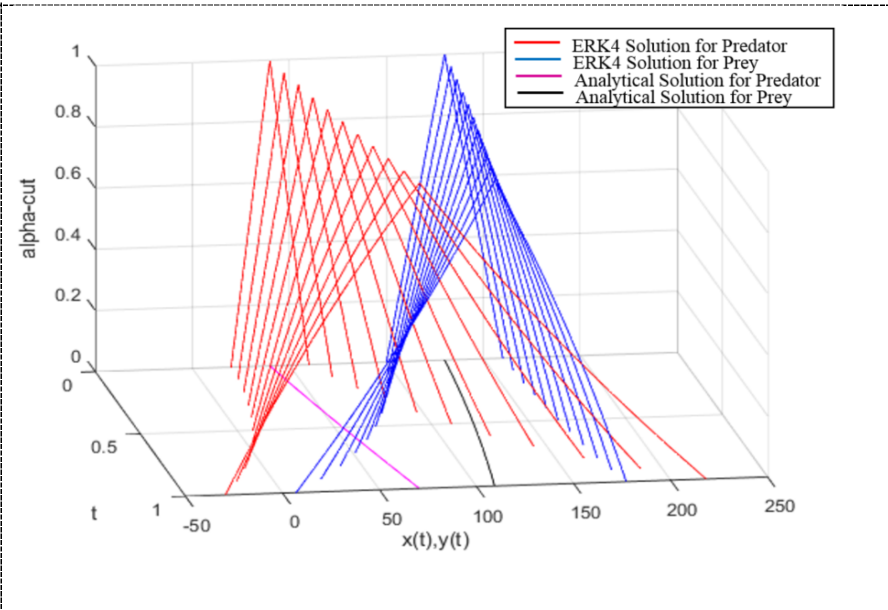

Fig. 3. Numerical result of interpolation ERK4 in 3D FPPS

5. Conclusions

In this paper, ERK4 method with polynomial interpolation technique was applied to find the numerical solution of FPPS. Figure 2 shows the plot of the crisp solution and fuzzy solution for $\alpha = 0$. Meanwhile, Figure 3 shows the crisp solution confined by the lower and upper bounds of the dependent variable $x(t)$, $y(t)$ when $\alpha = 0$. Additionally, when $\alpha = 1$ the projection of the peaks of the triangle coincides with the crisp solution. Figure 2 and Figure 3 indicates that the semi exact solution for predator prey are bounded in the fuzzy range of fuzzy solution when the semi exact solution at the middle between ERK4 lower solution and ERK4 upper solution, respectively. It is concluded that the current crisp fuzzy solution is in good agreement with semi exact solution. In other word, the technique of calculating semi exact solution is acceptable and validated.

References

[1] Zadeh, Lotfi Asker. "Fuzzy sets." *Information and control* 8, no. 3 (1965): 338-353.https://doi.org/10.1016/S0019-9958(65)90241-X

[2] Lixandru-Petre, Irina-Oana. "A fuzzy system approach for diabetes classification." In *2020 International Conference on e-Health and Bioengineering (EHB)*, pp. 1-4. IEEE, 2020. https://doi.org/10.1109/EHB50910.2020.9279882

[3] Gupta, Nisha, Harjeet Singh, and Jimmy Singla. "Fuzzy logic-based systems for medical diagnosis–A review." In *2022 3rd International Conference on Electronics and Sustainable Communication Systems (ICESC)*, pp. 1058-1062. IEEE, 2022. https://doi.org/10.1109/ICESC54411.2022.9885338

[4] Gumah, G., M. F. M. Naser, M. Al-Smadi, S. K. Q. Al-Omari, and Dumitru Baleanu. "Numerical solutions of hybrid fuzzy differential equations in a Hilbert space." *Applied Numerical Mathematics* 151 (2020): 402-412.https://doi.org/10.1016/j.apnum.2020.01.008

[5] Ullah, Abd, Aman Ullah, Shabir Ahmad, Imtiaz Ahmad, and Ali Akgül. "On solutions of fuzzy fractional order complex population dynamical model." *Numerical Methods for Partial Differential Equations* 39, no. 6 (2023): 4595-4615. https://doi.org/10.1002/num.22654

[6] Mechee, Mohammed S., and Ahlam A. Mahdi. "A Numerical Study for Solving Mathematical Fuzzy Population Model." In *Journal of Physics: Conference Series*, vol. 1999, no. 1, p. 012109. IOP Publishing, 2021.https://doi.org/10.1088/1742-6596/1999/1/012109

[7] Al-Smadi, Mohammed, Hemen Dutta, Shatha Hasan, and Shaher Momani. "On numerical approximation of Atangana-Baleanu-Caputo fractional integro-differential equations under uncertainty in Hilbert Space." *Mathematical Modelling of Natural Phenomena* 16 (2021): 41. https://doi.org/10.1051/mmnp/2021030

[8] Dong, Nguyen Phuong, Hoang Viet Long, and Nguyen Long Giang. "The fuzzy fractional SIQR model of computer virus propagation in wireless sensor network using Caputo Atangana–Baleanu derivatives." *Fuzzy Sets and Systems* 429 (2022): 28-59. https://doi.org/10.1016/j.fss.2021.04.012
[9] De Baets, Muhammad Zaini Ahmad Bernard. "A predator-prey model with fuzzy initial populations." (2009).
[10] Akın, Ömer, and Ömer Oruc. "A prey predator model with fuzzy initial values." *Hacettepe Journal of Mathematics and Statistics* 41, no. 3 (2012): 387-395.
[11] Pandit, Purnima, and Payal Singh. "Prey predator model with fuzzy initial conditions." *International Journal of Engineering and Innovative Technology (IJEIT) Volume* 3 (2014).
[12] Ahmad, Shabir, Aman Ullah, and Ali Akgul. "Study of a fractional system of predator-prey with uncertain initial conditions." *Mathematical Problems in Engineering* 2022 (2022). https://doi.org/10.1155/2022/3196608
[13] Sukarsih, I., A. K. Supriatna, E. Carnia, and N. Anggriani. "A Runge-Kutta numerical scheme applied in solving predator-prey fuzzy model with Holling type II functional response." *Frontiers in Applied Mathematics and Statistics* 9 (2023): 1096167. https://doi.org/10.3389/fams.2023.1096167
[14] Lambert, John Denholm. *Numerical methods for ordinary differential systems*. Vol. 146. New York: Wiley, 1991. http://ndl.ethernet.edu.et/bitstream/123456789/11075/1/106.pdf
[15] Wu, Xinyuan. "Two-step Runge–Kutta methods of order over five with reduced function evaluations." *Applied mathematics and computation* 155, no. 3 (2004): 779-811. https://doi.org/10.1016/s0096-3003(03)00815-4
[16] Rabiei, F., and F. Ismail. "New Improved Runge-Kutta Method with Reducing Number of Function Evaluations." In *International Conference on Software Technology and Engineering, 3rd (ICSTE 2011). ASME Press*. 2011.https://doi.org/10.1115/1.859797.paper14
[17] Ahmadian, Ali, Soheil Salahshour, and Chee Seng Chan. "A Runge–Kutta method with reduced number of function evaluations to solve hybrid fuzzy differential equations." *Soft Computing* 19, no. 4 (2015): 1051-1062. https://doi.org/10.1007/s00500-014-1314-9
[18] Musyaffafi, Ulia Maulidah, Mohammad Hafiyusholeh, Yuniar Farida, Aris Fanani, and Ahmad Zaenal Arifin. "Stability Analysis of Cycle Business Investment Saving-Liquidity Money (IS-LM) Model using Runge-Kutta Fifth Order and Extended Runge-Kutta Method." *BEST ICON* (2018): 240-246. https://doi.org/10.5220/0008905402400246
[19] Rangkuti, Yulita Molliq, and Akmal Huda. "Accuracy of Fifth-Order Improved Runge-Kutta Method for Handling Hyperchaotics Finance Systems." In *Journal of Physics: Conference Series*, vol. 1462, no. 1, p. 012025. IOP Publishing, 2020. https://doi.org/10.1088/1742-6596/1462/1/012025
[20] Goeken, David, and Olin Johnson. "Runge–Kutta with higher order derivative approximations." *Applied numerical mathematics* 34, no. 2-3 (2000): 207-218. https://doi.org/10.1016/S0168-9274(99)00128-2
[21] Wu, Xinyuan, and Jianlin Xia. "Extended Runge–Kutta-like formulae." *Applied numerical mathematics* 56, no. 12 (2006): 1584-1605. https://doi.org/10.1016/j.apnum.2005.11.008
[22] Zulkefli, Nor Atirah Izzah, Normah Maan, Yeak Su Hoe, and Nor Afifah Hanim Zulkefli. "Fuzzy logistic equation by polynomial interpolation." In *AIP Conference Proceedings*, vol. 1750, no. 1. AIP Publishing, 2016. https://doi.org/10.1063/1.4954573