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Efficient Homotopy Technique for the Computation of Bratu Differential Equation

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ABSTRACT

In this paper, we adopt a variant of the Homotopy Analysis Method (HAM) for the numerical solutions of the Bratu differential equation. A predictor-corrector approach essentially tracks the solutions of a homotopy path by using small steps together with a tangent vector derived from the system to determine a new point by a refinement of the previous point. An improved Euler step acts as a predictor along the track, followed by a Newton corrector. The procedure which involves the solution of systems of nonlinear differential equations gradually moves along the tangent of a solution path and subsequently arrives at a path of acceptable solutions of the target system. An attractive feature of this approach is the improvement in convergence resulting from the achievement of a global quadratic rate of convergence. Numerical results are compared with closed form solutions taken from literature. From these results, the method can be relied upon for applications to resolve complicated nonlinearities arising from problems in mathematical physics and engineering.

1. Introduction

Nonlinear differential equations are often employed to model real-world problems in many branches of science and engineering. Because of the mathematical rigor that accompany their formulations, analytical or closed form solutions are limited to very few examples. And for this reason, numerical and semi analytical solutions have vastly taken center stage in the past few decades. These include but are not limited to perturbation, homotopy, finite difference, Adomian decomposition, spectral element and collocation techniques [1-8]. Their applications have been deployed to better understand the factors that affect our lives and the environment as well as to better address the impact of modern technology on the society. As a result of this, and a concomitant improvement in computer technology, numerical methods are increasingly being employed to formulate and predict real-world challenges. There exists a continuous demand for novel methods of

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handling systems of nonlinear differential equations by improving on the accuracy, and replacing the differential operators with similar algebraic equations.

This encouraging scenario, has resulted in a better and efficient handling of systems of equations with totally filled coefficient matrices that are not limited to diagonal dominance. Iterative techniques are often employed to deal with such problems, and there is quite a lot of those for solving algebraic and transcendental equations, yet there still exists some serious numerical challenges accompanying such attempts in cases where convergence is local [9]. On the other hand, the use of perturbation techniques, though quite common and straightforward to implement requires the governing differential equations to contain large or small physical factors known as perturbation quantities. Unfortunately not all of the do. As a result the method can only handle equations that exhibit weak nonlinearity. In addition a lot of work still needs to be done to translate some of the excellent techniques adopted for the numerical solution of nonlinear algebraic equations to their nonlinear differential equation equivalents especially those that involve convergence issues arising from the constraints of the 'initial guess' [10-12]. The motivation of the current work seeks to address this challenge by further expanding the homotopy analysis method (HAM) to deal with a strongly nonlinear differential equation. Calculation of errors and comparison of results obtained herein with those found in literature are major components of this work.

The Homotopy analysis method (HAM) proposed by Liao [13-16] is a semi analytical technique that effectively handles nonlinearity . Unlike the perturbation technique, it is not limited by the presence and size of any physical parameter that comes with the differential equation. And more importantly it provides a simple way of assuring convergence . For these reasons, HAM has become a very reliable tool for handling strongly nonlinear equations irrespective of the first guess.

In this paper, we adopt a hybrid HAM formulation which includes the Newton and improved Euler formulations. Both these methods leverage on the concept of homotopy to convert a nonlinear differential equation into a composite of simpler ones that are solved by combining the strengths of the Newton and the Euler techniques. This hybrid approach handles the limitations of each technique holistically. Since HAM is based on topology's homotopy, it provides a a wide range of choice for selecting the 'first guess' . The Euler approach further refines and 'predicts' the this solution path which becomes available for 'correction' by the Newton method.

2. Problem Formulation

Homotopy techniques belong to a family of continuation methods. They are based on the idea of creating a new problem which is simpler than the original one and whose solutions are known and then gradually proceeding towards the system of equations whose solutions are sought. Let us take into consideration a system of **n** nonlinear equations. variables. This can be defined as:

$$\begin{cases} f_1\left(s_1 \quad s_2 \quad . \quad s_n\right)=0 \\ f_2\left(s_1 \quad s_2 \quad . \quad s_n\right)=0 \\ \qquad\qquad . \\ \qquad\qquad . \\ f_n\left(s_1 \quad s_2 \quad . \quad s_n\right)=0 \end{cases} \tag{1}$$

Another way to define equation (1) is to express it as a vector function $\mathbf{F}$ from $\Re^n \ to \ \Re^n$ i.e.

$$\mathbf{F}(s)=\left(f_1(s), \quad f_2(s) \quad f_3(s) \quad .. \quad f_n(s)\right)^T \tag{2}$$

where .

$$s = \begin{pmatrix} s_1 & s_2 & s_3 & . & . & . & s_n \end{pmatrix}^T$$

Equation (1) finally takes the form:

$$\mathbf{F}(\mathbf{s}) = \mathbf{0} \qquad (3)$$

Equation (3) can be a system of nonlinear transcendental, algebraic or differential equations. We recall that the Newton's method of solving such a system is given as:

$$\mathbf{s}^{(\mathbf{k+1})} = \mathbf{s}^{(\mathbf{k})} - \left[\mathbf{J}\left(\mathbf{s}^{(\mathbf{k})}\right)\right]^{-1} \mathbf{F}\left(\mathbf{s}^{(\mathbf{k})}\right), \quad k = 0,1,2....n \qquad (4)$$

where $\mathbf{J}\left(\mathbf{s}^{\kappa}\right)$ is the Jacobian matrix of $\mathbf{F}\left(\mathbf{s}^{(\mathbf{k})}\right)$ and $\mathbf{s}^{(\mathbf{0})}$ is an initial guess of the solution profile. The Jacobian function of this system is given as

$$\frac{\partial f_i}{\partial s_j} = \begin{bmatrix} \frac{\partial f_1}{\partial s_1} & \frac{\partial f_1}{\partial s_2} & . & . & . & \frac{\partial f_1}{\partial s_n} \\ \frac{\partial f_2}{\partial s_1} & \frac{\partial f_2}{\partial s_2} & . & . & . & \frac{\partial f_2}{\partial s_n} \\ \frac{\partial f_3}{\partial s_1} & \frac{\partial f_3}{\partial s_1} & . & . & . & \frac{\partial f_3}{\partial s_n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \frac{\partial f_n}{\partial s_1} & \frac{\partial f_n}{\partial s_2} & . & . & . & \frac{\partial fn}{\partial s_n} \end{bmatrix} \qquad (5)$$

Direct application of the Newton's method requires that the Jacobian matrix $\mathbf{F}'(\mathbf{s})$ should not be singular for the system to be solved. In addition to this constraint the first guess should be close to the root. In order to obviate these constraints, an auxiliary homotopy function is introduced. We consider a homotopy function:

$$H : [0,1] \times \Re^n \to \Re^n .... \qquad (6)$$

which is defined as :

$$H(s,q) = qF(s) + (1-q)\left(F(s) - F\left(s^{(0)}\right)\right) \qquad (7)$$

where $s^{(0)}$ is the initial guess of the iterative process, and $q \in [0,1]$ is an arbitrary homotopy parameter. We now have two boundary conditions namely:

$$H(s,0) = \left(F(s) - F\left(s^0\right)\right), \qquad H(s,1) = F(s) \qquad (8)$$

Continuation approach avoids the divergence problem by solving $H(s,q)=0$ instead of $\mathbf{F}(\mathbf{s})=\mathbf{0}$. Hence solving the system for $q\in(0,1)$ with the Newton's method, yields a family of H vs. q or solution curves defined by equation (8) which led from a known or 'guess value' $\mathbf{s}^{(0)}$ to an unknown solution $\mathbf{s}^{*}$. The desired solutions for the system can therefore be obtained by solving the following system of equations

$$\boldsymbol{\varphi}'(\mathbf{q})=-\left[\mathbf{J}(\boldsymbol{\varphi}(\mathbf{q}))\right]^{-1}\mathbf{F}(\boldsymbol{\varphi}(\mathbf{0}))\qquad 0\leq\leq q\leq 1 \tag{9}$$

where the initial condition is given as $\boldsymbol{\varphi}(\mathbf{0})=\mathbf{s}^{(0)}$ and $\mathbf{J}(\boldsymbol{\varphi}(\mathbf{q}))$ is the Jacobian matrix of the function H with respect to $\mathbf{s}$. Newton homotopy equation is defined as :

$$\mathbf{s}^{(\mathbf{k}+1)}=\mathbf{s}^{(\mathbf{k})}-\frac{\mathbf{H}\left(\mathbf{s}^{(\mathbf{k})},\mathbf{q}\right)}{\mathbf{H}'\left(\mathbf{s}^{(\mathbf{k})},\mathbf{q}\right)},\quad k=0,1,2.... \tag{10}$$

where $H'(s,q)=0$, is given as:

$$\mathbf{H}'(\mathbf{s},\mathbf{q})=(\mathbf{g})\mathbf{F}'(\mathbf{s})+(\mathbf{1}-\mathbf{g})\left(\mathbf{F}(\mathbf{s})-\mathbf{F}(\mathbf{s}^{\mathbf{0}})\right)'=0 \tag{11}$$

To avoid divergence

$$\mathbf{H}'(\mathbf{s},\mathbf{q})\neq(\mathbf{g})\mathbf{F}'(\mathbf{s})+(\mathbf{1}-\mathbf{g})\left(\mathbf{F}(\mathbf{s})-\mathbf{F}(\mathbf{s}^{\mathbf{0}})\right)'\neq 0 \tag{12}$$

Hence a unique solution can be obtained for $\mathbf{H}(\mathbf{s},\mathbf{q})=\mathbf{0}$

By chain rule:

$$\mathbf{H}'(s,q)=\frac{\partial\mathbf{H}}{\partial s}\frac{ds}{dq}+\frac{\partial\mathbf{H}}{\partial q}=0 \tag{13}$$

or

$$\frac{ds}{dq}=-\frac{\frac{\partial\mathbf{H}}{\partial q}}{\frac{\partial\mathbf{H}}{\partial s}}=-\frac{\mathbf{H}_q}{\mathbf{H}_s} \tag{14}$$

Equation (13) is the Newton Homotopy Differential Equation (NHDE) ,a first order ordinary differential equation that permits several numerical solution techniques. If for example we employ the Euler method, it can be recast as:

$$\mathbf{s}^{(k+1)}=\mathbf{s}^{(k)}+h\left(-\frac{\mathbf{H}_q}{\mathbf{H}_s}\right);\quad k=0,\,1,\,2...... \tag{15}$$

What equation (15) does essentially is to provide 's values' that not only avoid divergence, but also relax the constraint requiring initial guesses to be close to the vicinity of the actual solution. Hence, we are able to arrive at a numerical methodology that is more effective in dealing with complex problems with strong nonlinearities. In the work reported herein, we improve on this hybrid

Newton -Improved Euler -homotopy method and compare the quality of the solutions with those obtained in literature. We adopt the following computation steps:

INPUT PARAMETERS:

i. L: length of problem domain
ii. N_d : numberof nodes in problem domain
iii. u_0 : initial guess
iv. n: number of nodes in the problem domain
v. Maxiter: maxium number of iterations
vi. h: number of spaces between nodes
vii. dL: step length for homotopy parameter
viii. N: number of Euler steps

COMPUTATION STEPS:

i. Finite difference functional definition of governing differential equation (Problem specific):
$f_i = u_{i-1} - 2u_i + u_{i+1} + \frac{F}{h^2}$.
ii. Compute Jacobian matrix
iii. Compute NHDE dxdL using $dxdL = \frac{dx}{dL} = -J \setminus F$ (this essentially is the Euler slope)
iv. Apply the Euler method
v. $u^{(i+1)} = u^{(i)} + h\left(avg - dxdL\right)$
vi. Display Results

3. Numerical Experiments

Numerical solutions based on the Bratu-like equations are presented to verify the utility of this approach for highly nonlinear problems. These are accompanied by discussions on the quality of the overall output based on norm size, convergence characteristics and CPU times. And where the analytic solutions are available, the absolute error is chosen as a means of determining the quality the numerical computations. All examples are solved with MATLAB.

3.1 Example 1

Consider the 1-D nonlinear Bratu equation

$$u''(x) + \lambda e^{u(x)} = 0; \qquad 0 \le x \le 1$$
$$u(0) = u(1) = 0 \qquad (16)$$

The finite difference (FD) functional characterization of equation (16) for each node in the problem domain is given as :

$$F(u_j) = fh^2 + u_{j-1} - 2u_j + u_{j+1} \quad for \quad j = 1, 2......N \qquad (17)$$

where $f = -\lambda e^{(u_j)}$

In order to guarantee the accuracy of the technique applied herein, we calculate the infinity norms generated by the homotopy function as the calculation proceeds. Fig. 1 displays the trend for various values of λ. An increase is observed, followed by a sharp decrease in magnitude up to $\lambda = 1.5$, after this the figure proceeds upwards for several values considered. This can be attributed to the instability encountered by the scheme in dealing with the bi-stability and subsequent bifurcation exhibited by the Bratu equation as λ approaches a certain critical value (this is outside the scope of this work).

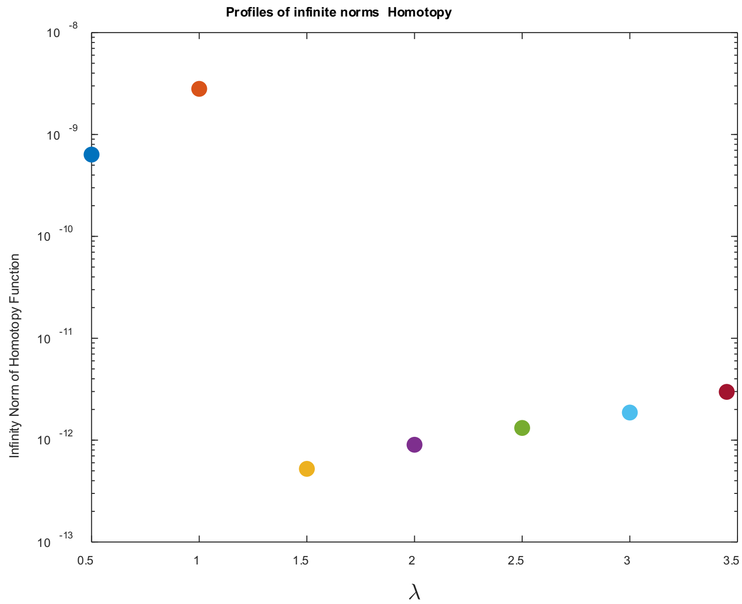

Fig. 1. Infinity norm of Homotopy function Vs. λ

3.2 Example 2

We further validate accuracy by determining the absolute errors between analytical solution of the Bratu equation [9] and that computed by the Homotopy method. In order to facilitate this procedure, we compare the mid-point values obtained from both techniques for different values of λ. Fig. 2 shows the profile of the absolute error magnitude for the two methods in response to the variations in λ. The result is encouraging and affirms that the numerical scheme can yield accurate results when applied to the computation of the Bratu equation.

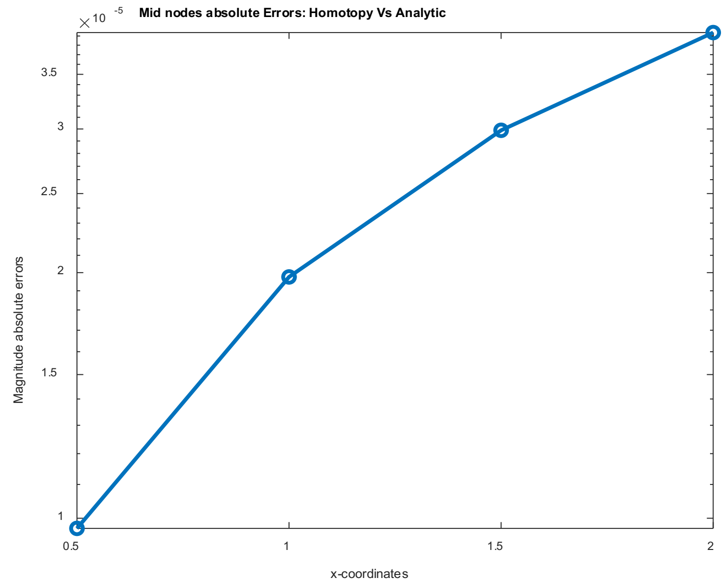

Fig. 2. Mid node absolute errors Vs λ

3.3 Example3

Now that the scheme can be trusted to yield accurate resuts, it is then applied to generate scalar profiles for different values of λ. Fig. 3 displays such profiled for $\lambda = 2,3$. It is worthwhile to note that as λ values increase the profile becomes steeper. This is confirmed by a 3-D display of the profiles for different values of λ (see Fig. 4).

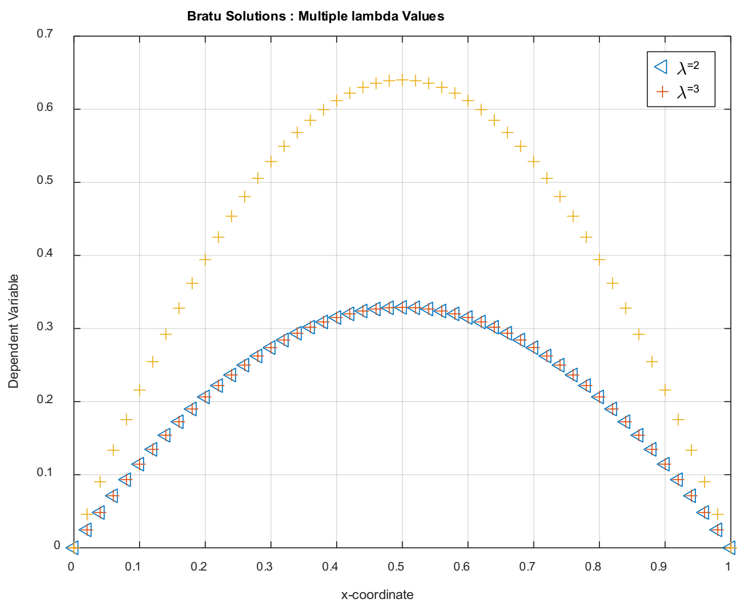

Fig. 3. Bratu equation solution profiles for multiple λ values

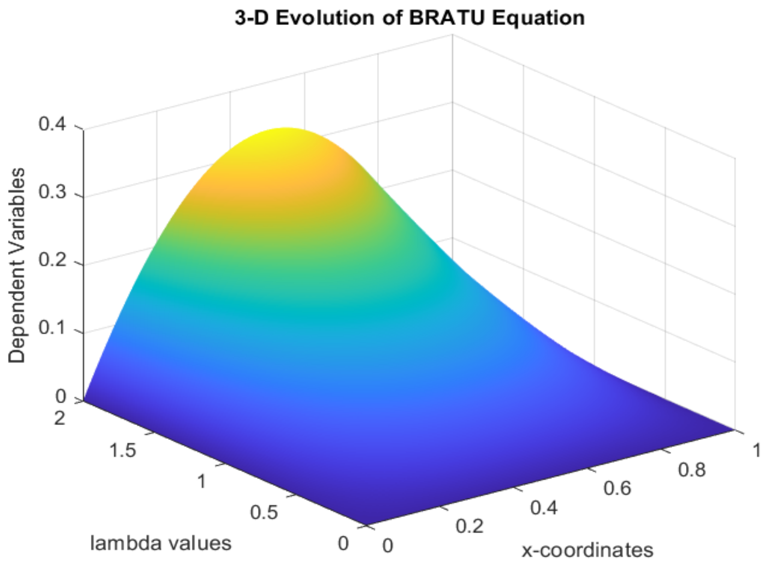

Fig. 4. A 3-D solution profiles for multiple λ values

We further put the efficacy of this technique to test for a critical λ $(\lambda_c = 3.45)$. It should be borne in mind that the it is extremely difficult for the Bratu equation to yield numerical solutions close to or in the vicinity of this value.

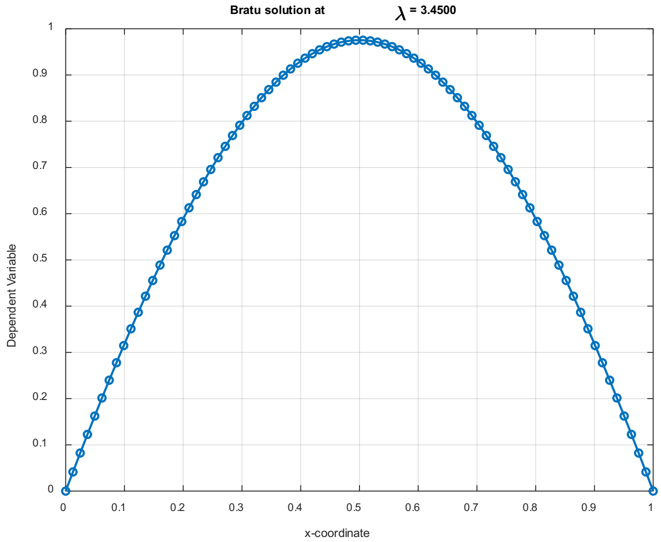

Fig. 5. Bratu equation solutionfor $\lambda = 3.45$

3.4 Example 4

Fig. 6 shows the path for mid node values of the solution profiles for various values of λ. The upward trajectory of the path together with an increase in gradient are i in total agreement with figures 3 and 4 of this paper.

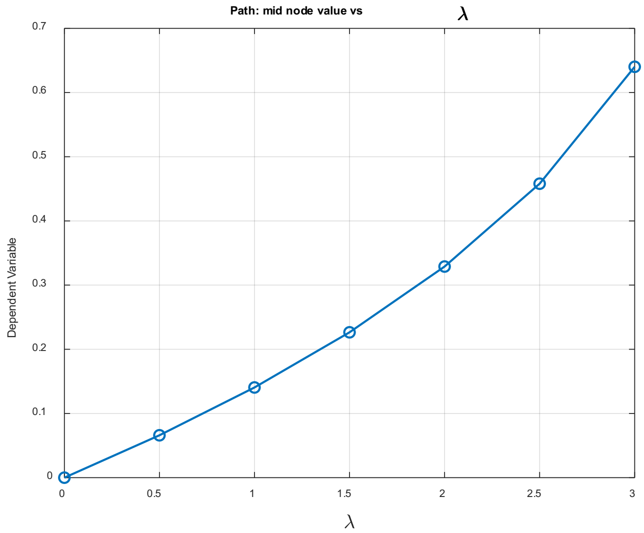

Fig. 6. Path for midpoint values of solution profiles for multiple λ values

4. Conclusions

In the work reported herein, Newton-Homotopy using a modified predictor-corrector approach, is applied to a highly nonlinear differential equation, namely the Bratu equation. Excellent convergent and accurate results are obtained for the examples tested. This includes results for relatively high values of λ which creates considerable numerical challenge for many traditional methods. Because of the encouraging start system embedded in the predictor-corrector approach our future work will involve adapting this same technique to admit relaxed versions of the Newton's method.

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The Author is the sole contributor to the contents of this paper

Data Availability Statement
All data generated or analyzed during this study are included in this published article. Additional datasets are available from the corresponding author upon reasonable request.

Ethics Statement
This study was conducted in accordance with the ethical standards of the institutional and national research committee.

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