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A Semi-Analytical Solution of a Two-Dimensional Moving Boundary Problem Using the Homotopy Analysis Method

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ABSTRACT

This study presents a semi-analytical solution to a two-dimensional moving boundary problem governed by the classical heat equation. The physical model represents heat conduction within a rectangular domain, where the boundary evolves dynamically over time due to phase change phenomena. A Stefan-type condition is imposed at the moving interface to capture the effect of latent heat exchange. The Homotopy Analysis Method (HAM) is employed to construct convergent series solutions for both the temperature field and the moving boundary function. Unlike standard applications of HAM where convergence control is achieved via an auxiliary parameter, the convergence of the series solution in this work is inherently guided by the imposed boundary conditions. This boundary-driven convergence ensures consistency between the evolving interface and the thermal field without requiring external tuning of convergence control parameters. The analytical results are compared with numerical benchmarks to validate the accuracy and reliability of the proposed approach. The study demonstrates that HAM provides a robust framework for analyzing two-dimensional moving boundary problems with analytically tractable and physically consistent solutions.

1. Introduction

Modeling heat transfer in systems undergoing phase change is a classical yet continually evolving area of applied mathematics and thermal physics. A prominent class of such problems is known as Stefan problems, where the domain boundary is not fixed but evolves over time due to latent heat exchange during phase transition [3,4]. These moving boundary problems arise naturally in numerous scientific and engineering contexts, including melting and solidification of materials, ice formation, thermal ablation, laser heating, and crystal growth [1,11]. Traditional approaches to solving Stefan problems often rely on numerical techniques such as finite difference, finite element, or front-tracking methods [2,10]. While powerful, these methods typically require discretization, introduce numerical diffusion, and often struggle with the accurate resolution of the moving boundary, especially in two-dimensional settings. Furthermore, they provide limited analytical insight into the structure of the solution and the interplay between the temperature field and the moving interface.

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In contrast, semi-analytical techniques offer a valuable alternative for exploring such problems, particularly when a closed-form understanding of the solution structure is desired. Among these, the Homotopy Analysis Method (HAM)—first developed by Liao [6]—has emerged as a flexible and systematic approach for handling nonlinear problems without relying on small parameters or linearization. HAM constructs a convergent series solution through an embedding process that continuously deforms a simple initial guess toward the true solution [7, 8]. In this work, we apply HAM to a two-dimensional moving boundary problem governed by the classical heat conduction equation. A Stefan-type condition is imposed at the moving interface to capture the thermal flux associated with the phase change. Unlike conventional HAM implementations, the convergence control parameter $\hbar$ is determined using the prescribed boundary conditions, rather than through the classical approach of minimizing the residual square of the original governing equation or using $\hbar$-level curves [13,14]. This boundary driven convergence ensures compatibility between the temperature solution and the evolving free boundary, leading to a physically consistent and analytically elegant solution framework [5,9,12,15].

2. Governing Equations

Let $\Omega = \{(x, y, t) \in \mathbb{R}^3 | 0 < x < 1, 0 < y < p(x,t), 0 < t < 1\}$ be the solution domain, where $p(x,t)$ is an unknown part of the boundary to be determined see Figure 1 for an illustration. The temperature distribution $u(x, y, t)$ is unknown and to be determined during the solution process. The mathematical formulation for a moving boundary problem is given by

$$u_t = u_{xx} + u_{yy}, 0 < x < 1, 0 < y < p(x,t), 0 < t < 1 \quad (1)$$

where the initial condition is

$$u(x, y, 0) = f(x, y),\ \ 0 < x < 1, 0 < y < p(x,t) \quad (2)$$

with the following boundary conditions

$$u_x(0, y, t) = a(y, t),\ \ 0 < y < p(x,t), 0 < t < 1 \quad (3)$$

$$u(x, p(x,t), t) = b(x, t),\ \ \ 0 < x < 1, 0 < t < 1 \quad (4)$$

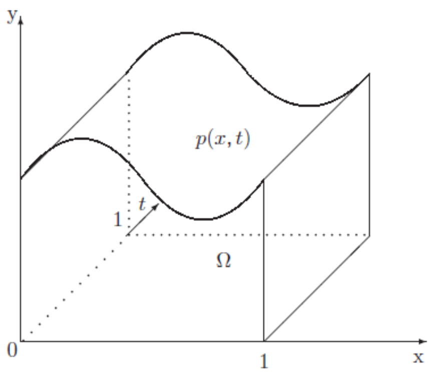

Fig. 1. Solution domain

3. Application of the Homotopy Analysis Method (HAM)

Consider the differential equation

$$\aleph[(\xi, t)] = \omega(\xi, t) \quad (5)$$

where $\xi \in (x, y)$. Using homotopy, a basic concept in topology

$$(1-q)\mathcal{L}[\varphi(\xi, t; q) - u_0(\xi, t)] = \hbar q \mathcal{H}(\xi, t)\aleph[\varphi(\xi, t; q) - \omega(\xi, t)] \quad (6)$$

where $\mathcal{L}$ is an auxiliary linear operator with the property

$$\mathcal{L}[0] = 0 \quad (7)$$

$\aleph$ - the nonlinear operator related to the original equation (5),

$q \in [0,1]$ - the embedding parameter in topology (called the homotopy parameter),

$\varphi(\xi, t; q)$ - the solution for equation (6) for $q \in [0,1]$,

$u_0(\xi, t) -$ the initial guess for $u(\xi, t)$,

$\hbar \neq 0$ - the convergence control parameter,

$\mathcal{H}(\xi, t)$ – an auxiliary function that is non – zero almost everywhere

When $q = 0$ due to the property $\mathcal{L}[0] = 0$, equation (2) becomes

$$\varphi(\xi, t; 0) = u_0(\xi, t) \quad (8)$$

When $q = 1$, with $\hbar \neq 0$ and $\mathcal{H}(\xi, t) \neq 0$ equation (2) becomes equivalent to the original nonlinear equation (5) so that we have

$$\varphi(\xi, t; 1) = u(\xi, t) \quad (9)$$

where $u(\xi, t)$ is the solution to equation (5). As the homotopy parameter q increases from 0 to 1, the solution $\varphi(\xi, t; q)$ of equation (6) varies (or deforms) continuously from the initial guess $u_0(\xi, t)$ to the solution $u(\xi, t)$ of the original equation (5). This is why equation (6) is called the *zeroth-order deformation* equation.

If $\mathcal{L}, \mathcal{H}(\xi, t)$ and $\hbar$ are properly chosen so that the solution $\varphi(\xi, t; q)$ of the *zeroth-order deformation* equation (6) always exists for $q \in [0,1]$ and it is analytic at $q = 0$; the Maclaurin series solution for $\varphi(\xi, t; q)$ with respect to q, i.e.

$$\varphi(\xi, t; q) = u_0(\xi, t) + \sum_{m=1}^{\infty} u_m(\xi, t) q^m \quad (10)$$

converges at $q = 1$. Then, due to equation (9), we have the approximation series

$$u(\xi, t) = u_0(\xi, t) + \sum_{m=1}^{\infty} u_m(\xi, t) \quad (11)$$

Substituting the series equation (10) into the *zeroth-order deformation* equation (6), we have the high-order approximation equations for $u_m(\xi, t)$ called the *mth-order deformation* equation

$$\mathcal{L}[u_m(\xi, t) - \chi_m u_{m-1}(\xi, t)] = \hbar \mathcal{H}(\xi, t) R_m(u_{m-1}(\xi, t)) \quad (12)$$

where

$$R_m(u_{m-1}(\xi,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[\frac{\partial \varphi(\xi,t;q)}{\partial t} - (\aleph[\varphi(\xi,t;q)] - \omega(\xi,t))\right]\Bigg|_{q=0} \quad (13)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1 \end{cases} \quad (14)$$

For our governing equations, equation (1) – equation (4)

$$R_m\big(u_{m-1}(\xi,t)\big) = \frac{\partial u_{m-1}(\xi,t)}{\partial t} - \frac{\partial u_{m-1}(\xi,t)}{\partial x^2} - \frac{\partial u_{m-1}(\xi,t)}{\partial y^2} \quad (15)$$

Now the solution of *mth-order deformation* equation (12) for $m \geq 1$ reads

$$u_m(\xi,t) = \chi_m u_{m-1}(\xi,t) + \hbar\mathcal{L}^{-1}[\mathcal{H}(\xi,t)R_m(u_{m-1}(\xi,t))] + c \quad (16)$$

where c is the integration constant which is determined by the initial condition $u_0(\xi,t)$.

Now from equation (16) the values $u_m(\xi,t)$ for $m = 1,2,3, \ldots$ can be obtained and the series solutions are thus gained. Finally, the approximate solution is gained by truncating the series as

$$u_m(\xi,t) = \sum_{i=0}^{m} u_i(\xi,t) \quad (17)$$

It is clear from equation (17) that $u_m(\xi,t)$ contains the convergence control parameter $\hbar$, which determines the convergence region and rate of the homotopy series solution.

Unlike the conventional Homotopy Analysis Method (HAM) where $\hbar$ is found by using $\hbar$ -level curves or by finding minimizing the residual square of the governing equation; we $\hbar$ by setting

$$(u_x(0,y,t))_{HAM} = a(y,t) \quad (18)$$

for every time step.

and the unknown boundary $p(x,t)$ is found by setting

$$(u(x,p(x,t),t))_{HAM} = b(x,t) \quad (19)$$

4. HAM Solutions to Some Examples

Consider equations (1) – (4), equation (18) and equation (19)

Example 1

$$f(x,y) = exp\left(\frac{x+y-2}{4}\right) - 1 \quad (20)$$
$$a(y,t) = \frac{1}{4}exp\left(\frac{t}{8} + \frac{y-2}{4}\right) \quad (21)$$
$$b(x,t) = 0 \quad (22)$$

With the Exact Solutions as

$$u(x,y,t) = exp\left(\frac{t}{8} + \frac{x+y-2}{4}\right) - 1 \text{ , } p(x,t) = 2 - x - \frac{1}{2}t \quad (23)$$

Using the methodology discussed in Section 3, we have the following error analysis for $u(x,t)$ and $p(x,t)$ when $x = 0.75$ and $y = 0.80$

$$Relative\ Error\ for\ u(x,y,t) = \left|\frac{(u(x,y,t))_{exact} - (u(x,y,t))_{HAM}}{(u(x,y,t))_{exact}}\right| \quad (24)$$

Table 1
Error analysis of $u(x,y,t)$ when $x = 0.75$ and $y = 0.80$

t	$\hbar$	$RE\ for\ u(x,y,t)$
0.1	-1.006276124	$9.562582066 \times 10^{-9}$
0.2	-1.012604819	$1.795153671 \times 10^{-7}$
0.3	-1.018986590	0.000001065509475
0.4	-1.025421928	0.000004078436894
0.5	-1.031911342	0.00001256044231
0.6	-1.038455346	0.00003503653303
0.7	-1.045054450	0.00009822970254
0.8	-1.051709181	0.0003381623359
0.9	-1.058420062	0.0000068316920
1.0	-1.065187625	0.0008404889568

$$Relative\ Error\ for\ p(x,t) = \left|\frac{(p(x,t))_{exact} - (p(x,t))_{HAM}}{(p(x,t))_{exact}}\right| \quad (25)$$

Table 2
Error analysis of $p(x,t)$ when $x = 0.75$

t	$\hbar$	$RE\ for\ p(x,t)$
0.1	-1.006276124	$3.333333333 \times 10^{-9}$
0.2	-1.012604819	$5.652173913 \times 10^{-8}$
0.3	-1.018986590	$3.027272727 \times 10^{-7}$
0.4	-1.025421928	0.000001002857143
0.5	-1.031911342	0.000002576000000
0.6	-1.038455346	0.000005637052632
0.7	-1.045054450	0.00001105100000
0.8	-1.051709181	0.00002001658824
0.9	-1.058420062	0.00003415800000
1.0	-1.065187625	0.00005568266667

Example 2

$$f(x,y) = sin\left(\frac{y-1}{\sqrt{2}}\right) sin\left(\frac{x+1}{\sqrt{2}}\right) \quad (26)$$

$$a(y,t) = \frac{1}{\sqrt{2}} sin\left(\frac{y-1}{\sqrt{2}}\right) cos\left(\frac{1}{\sqrt{2}}\right) \quad (27)$$

$$b(x,t) = 0 \quad (28)$$

Where the Exact Solutions are

$$u(x,y,t) = \exp(-t)\, sin\left(\frac{y-1}{\sqrt{2}}\right) sin\left(\frac{x+1}{\sqrt{2}}\right) \text{ , } p(x,t) = 1 \quad (29)$$

Using the methodology discussed in Section 3, we have the following error analysis for $u(x,t)$ and $p(x,t)$ when $x = 0.25$ and $y = 0.75$

Table 3
Error analysis of $u(x, y, t)$ when $x = 0.25$ and $y = 0.75$

t	$\hbar$	$RE\ for\ u(x, y, t)$
0.1	-0.9255635660	$1.625558975 \times 10^{-9}$
0.2	-0.8591393414	$8.982602517 \times 10^{-10}$
0.3	-0.7996214267	0
0.4	-0.7460911000	$5.485687747 \times 10^{-10}$
0.5	-0.6977793714	$1.576281867 \times 10^{-9}$
0.6	-0.6540380759	$9.380327803 \times 10^{-10}$
0.7	-0.6143173183	$1.184784627 \times 10^{-9}$
0.8	-0.5781478638	$1.309389515 \times 10^{-9}$
0.9	-0.5451270969	$1.266211810 \times 10^{-9}$
1.0	-0.5149078483	$2.199026453 \times 10^{-9}$

During the solution process,

$$(p(x, t))_{HAM} = 1 \quad (30)$$

Example 3

$$f(x, y) = \exp(x + y) \quad (31)$$
$$a(y, t) = \exp(y) \quad (32)$$
$$b(x, t) = xt \quad (33)$$

Where the Exact Solutions are

$$u(x, y, t) = \exp(2t + x + y)\ ,\ p(x, t) = -2t - x + \ln(xt) \quad (34)$$

Using the methodology discussed in Section 3, we have the following error analysis for $u(x, t)$ and $p(x, t)$

Table 4
Error analysis for $u(x, y, t)$ when $x = 0.58$ and $y = 0.27$

t	$\hbar$	$RE\ for\ u(x, y, t)$
0.01	-1.015962058	$1.466330422 \times 10^{-8}$
0.02	-1.032309928	$2.402336154 \times 10^{-7}$
0.03	-1.049055180	0.000001243799852
0.04	-1.066210820	0.000004024447845
0.05	-1.083791582	0.00001005758706
0.06	-1.101811114	0.00002135678020
0.07	-1.120284291	0.00004053270019
0.08	-1.139226508	0.00007086390154
0.09	-1.158654147	0.0001163721309
0.1	-1.178583924	0.0001819133892

Table 5
Error analysis for $p(x,t)$ when $x = 0.58$

t	$\hbar$	$RE\ for\ p(x,y)$
0.01	-1.015962058	$2.434820671 \times 10^{-9}$
0.02	-1.032309928	$4.727433721 \times 10^{-8}$
0.03	-1.049055180	$2.61725445 \times 10^{-7}$
0.04	-1.066210820	$9.096657182 \times 10^{-7}$
0.05	-1.083791582	0.000002382152858
0.06	-1.101811114	0.000005262758578
0.07	-1.120284291	0.00001032979921
0.08	-1.139226508	0.00001859777500
0.09	-1.158654147	0.00003134641989
0.1	-1.178583924	0.00005015531792

5. Analysis and Conclusion

The numerical results obtained using the Homotopy Analysis Method (HAM) for both the temperature distribution $u(x,y,t)$ and the moving boundary $p(x,t)$ show excellent agreement with known or exact solutions. A key innovation in this study was the use of the boundary condition $u_x(0,y,t)$ to compute the convergence control parameter $\hbar$ at each time step. This dynamic, boundary-driven approach led to significantly faster convergence, reduced computational time, and minimized relative error (RE).

Unlike earlier applications of HAM, which commonly determine $\hbar$ using heuristic $\hbar$-curves or by minimizing the average residual error of the discretized solution—often applying a single $\hbar$ value across all time steps—our method adapts $\hbar$ at each time level based on physical boundary data. This customization enhances the accuracy and efficiency of the solution process.

This study presents a novel and elegant approach for determining the convergence control parameter $\hbar$, one that can be extended to a wide range of linear and nonlinear ordinary and partial differential equations. We have demonstrated that HAM is not only suitable for one-dimensional problems but is also a powerful and flexible tool for solving complex two-dimensional moving boundary problems with strong physical and mathematical consistency.

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Conflict of Interest Statement

The authors declare no conflict of interest related to the publication of this work. No financial support, sponsorship, or other forms of compensation were received that could influence the results or interpretations presented. The study was conducted independently and in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

References

[1] Worster, Grae. "Mathematical Modeling of Melting and Freezing Processes. By V. ALEXIADES and AD SOLOMON. Taylor & Francis, 1993. 323 pp.£ 35." *Journal of Fluid Mechanics* 251 (1993): 719-720. https://doi.org/10.1017/S0022112093213593

[2] Buzdov, B. K. "On a two-dimensional boundary-value Stefan-type problem arising in cryosurgery." *Journal of Mathematical Sciences* 260, no. 3 (2022): 294-299. https://doi.org/10.1007/s10958-022-05693-3

[3] Chaurasiya, Vikas, Subrahamanyam Upadhyay, Kabindra Nath Rai, and Jitendra Singh. "A new look in heat balance integral method to a two-dimensional Stefan problem with convection." *Numerical Heat Transfer, Part A: Applications* 82, no. 9 (2022): 529-542. https://doi.org/10.1080/10407782.2022.2079829

[4] Crank, John. *Free and Moving Boundary Problems*. Oxford: Clarendon Press, 1984.

[5] Gammon, J., and J. A. Howarth. "The two-dimensional stefan problem with slightly varying heat flux." *International communications in heat and mass transfer* 22, no. 5 (1995): 629-638. https://doi.org/10.1016/0735-1933(95)00049-5
[6] Liao, Shijun. *Beyond perturbation: introduction to the homotopy analysis method*. Chapman and Hall/CRC, 2003.
[7] Liao, Shijun. *Homotopy analysis method in nonlinear differential equations*. Vol. 153. Beijing: Higher education press, 2012. https://doi.org/10.1007/978-3-642-25132-0
[8] Liao, Shijun, ed. *Advances in the Homotopy Analysis Method*. Singapore: World Scientific, 2014. https://doi.org/10.1142/8939
[9] Liu, Ji-Chuan, and Ting Wei. "Moving boundary identification for a two-dimensional inverse heat conduction problem." *Inverse Problems in Science and Engineering* 19, no. 8 (2011): 1139-1154. https://doi.org/10.1080/17415977.2011.603084
[10] Mahalakshmi, M., R. Rajaraman, G. Hariharan, and K. Kannan. "Approximate analytical solutions of two dimensional transient heat conduction equations." *Applied Mathematical Sciences* 6, no. 71 (2012): 3507-3518.
[11] Meirmanov, A. M. *The Stefan Problem*. De Gruyter Expositions in Mathematics 3. Berlin: Walter de Gruyter, 1992.
[12] Mochnacki, B., S. Lara, and E. Pawlak. "Numerical Solution of 2D Stefan Problem." *Solidification of Metals and Alloys* 2, no. 44 (2000). https://www.researchgate.net/publication/268272857_NUMERICAL_SOLUTION_OF_2D_STEFAN_PROBLEM.
[13] Rodrigues, José Francisco. "Mathematical models for phase change problems." In *Proc. European Workshop held at Obidos, Portugal*, vol. 88. 1988.
[14] Rubinshteĭn, L. I. *The stefan problem*. Vol. 27. American Mathematical Soc., 1971.
[15] Segal, Guus, Kees Vuik, and Fred Vermolen. "A conserving discretization for the free boundary in a two-dimensional Stefan problem." *Journal of Computational Physics* 141, no. 1 (1998): 1-21. https://doi.org/10.1006/jcph.1998.5900