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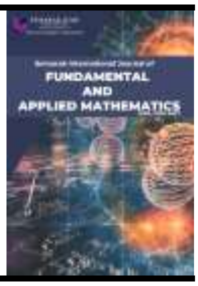


# Analytical Solution for an Inhomogeneous Heat Equation with Dirichlet Boundary Conditions

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**ABSTRACT**

In the subject of temperature distribution in a rod, the problem that is dealt with in the literature is one in which heat is transferred only through the cross section of a rod, without heat sources and with homogeneous boundary conditions. In this article, the one-dimensional heat equation is studied, considering heat transfer by convection on the lateral surface of a rod, Dirichlet boundary conditions and an initial arbitrary temperature distribution. This heat equation is derived from Fourier's law of conduction, the conservation of thermal energy, and Newton's law of cooling. A variant of an analytical method was used to find a solution, which meets the established initial condition and the boundary conditions. In the case of the initial condition, the convergence is not good at the ends of the rod due to the Gibbs phenomenon in the Fourier series. It is proved that such a solution can model the solution of less general problems. The solution found, contains a discontinuity when the convective thermal conductance is equal to zero, but taking the limit at this value, the function converges to the analytical solution where the convection factor is equal to zero.

## 1. Introduction

The study of the theory of energy conduction by heat leads to a Partial Derivative Equation (PDE), known as the diffusion equation. When heat conduction is restricted to a spatial dimension, the diffusion equation is called the "heat equation" or one-dimensional heat equation. This equation plays an important role in a wide range of practical applications in different areas, such as thermodynamics and fluid mechanics, etc [1-4].

The equation that is addressed in this article is the heat Eq. (1) with its boundary condition in the Eq. (2) (Dirichlet conditions) and its initial condition in the Eq. (3), which is a non-homogeneous linear PDE with boundary conditions also inhomogeneous, which makes it difficult to use general solution methods. This equation has a more practical meaning than the classical equation, since being $H \neq 0$,

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convection is considered within the heat transfer process. A heat transmission mechanism that is generally considered is radiation [5], however in this work it is assumed that the rod is perfectly reflective on its surface.

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} - H(T - T_a),\;\; 0 < x < L\,, t > 0 \tag{1}$$

$$T(0,t) = T_A\,, T(L,t) = T_B,\;\; t > 0 \tag{2}$$

$$T(x,0) = f(x), 0 < x < L \tag{3}$$

The inhomogeneous heat equation is commonly studied in university textbooks on differential equations and advanced mathematics, for this reason it is important to verify that Eq. (1) with its conditions (2) has not been solved. In [6,7] the Eq. (4) with its conditions (5) and (6) is proposed and the solution in stable state is requested, that is, when $t \to \infty$, that in this case the solution is Eq. (7). The approach to this problem with values at the boundary (BVP) is similar to the BVP of equation (1), with the difference that $L = 1, T(L,t) = 0$, but what is asked is not its general solution. Also, these references the BVP for Eq. (8) is proposed, where $T_a = 0$ y $T(0,t) = 0$, constituting a particular case of the BVP of Eq. (1) with the corresponding conditions, see Eqs. (9) and (10).

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} - H(T - T_a)\;\;,\;\; 0 < x < L\,, t > 0 \tag{4}$$

$$T(0,t) = T_a\;\;, T(1,t) = 0,\quad t > 0 \tag{5}$$

$$T(x,0) = f(x), 0 < x < 1 \tag{6}$$

$$\psi(x) = T_a\left(1 - \frac{sinh\sqrt{H/D}x}{sinh\sqrt{H/D}}\right) \tag{7}$$

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} - HT\;\;,\;\; 0 < x < \pi\,, t > 0 \tag{8}$$

$$T(0,t) = 0, T(1,t) = T_a,\quad t > 0 \tag{9}$$

$$T(x,0) = f(x), 0 < x < \pi \tag{10}$$

The method of separation of variables is used directly when both the partial derivative equation and the boundary conditions are homogeneous [8]. The boundary conditions with prescribed temperature can be temporary or non-temporary, in this article they are considered non-temporary. In [9], the BVP of Eq. (1) is not stated. In [10] the problem proposed in this article is presented in the section "Some important partial differential equations". Conditions (2) and (3) are stated and the deduction of Eq. (1) is assigned, but the solution of the PDE is not requested nor is it indicated how to solve it.

In [11], the Eq. (1) is not proposed, the closest that is done is to propose the substitution of $T(x,t) = \phi(x,t)e^{-Ht}$ where $H$ is a constant, to reduce Eq. (1) with $T_a = 0$ to a homogeneous heat equation.

A search was performed using the keyword Heat equation on google scholar and no analysis or solution for this equation was found. Science Direct searched the 2010-2024 interval with the words

"Heat equation", "inhomogeneous heat equation" in the area of "Find articles with these terms”. The “Title” field was also placed with the phrase “An inhomogeneous heat equation with inhomogeneous boundary conditions: an analytical solution”. About 28 articles containing the keywords were found, but the approach or solution of Eq. (1) was not found. Al-Nuaimi *et al.,* [12] performed a study on the transient temperature response of a cracked plate under thermal shock; with the methodology they proposed, partial derivation was avoided as well as discontinuity problems. Kovács [13] carried out a review of the equations used to analyze heat transfer in models not based on the law of heat conduction.

### *1.1 Statement of the Problem and Physical Interpretation*

In the Eq. (1), the term $H(T-T_a)$ represents convective heat transfer between a surface and a surrounding fluid [14], where H is the convective thermal conductance in W/K or W/°C, $T$ is the temperature on the surface of the rod and $T_a$ is the temperature of the surrounding fluid or ambient temperature, both in K or°C. The term $\partial^2 T/\partial x^2$ represents the extent to which the material surrounding a point is hotter or colder than the point on the rod [1]. According to the second law of thermodynamics, energy flows from bodies with a higher temperature to those with a lower temperature, so the temperature at a point x on the rod will vary at a rate ∂T⁄∂t which will depend on the fact that so cold or hot (expression $D(\partial^2 T/\partial x^2)$) and now the term $H(T-T_a)$) is also considered is the surrounding medium, in Figure 1 a diagram is shown physics of the heat flow in the thin rod. The derivation of Eq. (1) is based on the law of conservation of energy, Fourier's law of conduction, and Newton's law of cooling; its deduction without considering convection can be found in different sources [7,8]. The deduction of Eq. (1) is shown in the section 2.2.

The main contributions of this work are:

i) Find a function of the temperature $\boldsymbol{T(x,t)}$ that satisfies the conditions (1), (2) and (3).
ii) Analyze how the solution found can model less general problems posed in the literature.

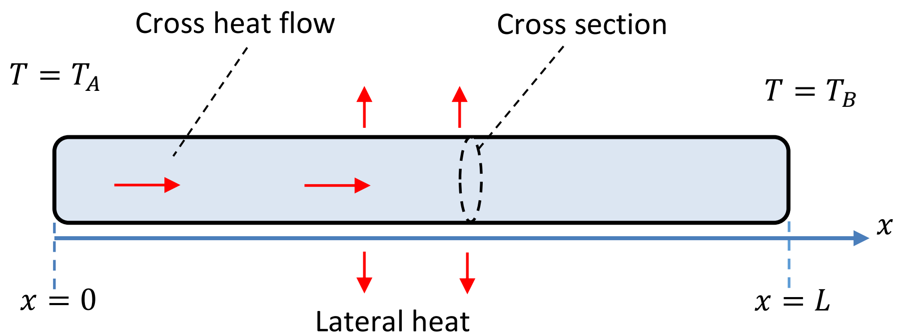


**Fig. 1.** Heat flow in a through a cross section *A* in an interval of length *L*

## 2. Methodology and problem solving

The method used to solve Eq. (1), is a variant of an analytical method presented by Zill and Gullen [6], which is used to solve non-homogeneous equations or with non-homogeneous boundary conditions, such as the one shown in shown in Eq. (11) with its conditions (12) and (13). This method consists of applying the substitution $T_1(x,t) = v_1(x,t) + \psi_1(x)$ to reduce the original problem to two problems, problems A and B as can be seen below.

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} \ , \ 0 < x < L\,, t > 0 \tag{11}$$

$$T(0,t) = T_A \ , T(L,t) = T_B,\ t > 0 \tag{12}$$

$$T(x,0) = f(x), 0 < x < L \tag{13}$$

Problem A:

$$\begin{cases} k\,\psi_1{}'' + F(x) = 0 \\ \psi_1(0) = T_A \\ \psi_1(0) = T_B \end{cases}$$

Problem B:

$$\begin{cases} \frac{\partial \mathrm{v}_1}{\partial \mathrm{t}} = \mathrm{D}\frac{\partial^2 \mathrm{v}_1}{\partial \mathrm{x}^2} \\ \mathrm{v}_1(0,\mathrm{t}) = 0, \mathrm{v}_1(\mathrm{L},\mathrm{t}) = 0 \\ \mathrm{v}_1(\mathrm{x},0) = \mathrm{f}(\mathrm{x}) - \psi(\mathrm{x}) \end{cases}$$

When developing the method of [6], it is found that the solution of Eq. (11) and that satisfies the conditions (12) and (13) is

$$T_1(x,t) = V_1(x,t) + \psi_1(x) = \sum_{n=1}^{\infty} A_n e^{-D\lambda^2 t} sen\left(\frac{n\pi}{L}x\right) + \left(\frac{T_B - T_A}{L}\right)x + T_A \tag{14}$$

where

$$A_n = \frac{2}{L}\int_0^L [f(x) - \psi(x)]\,sen\frac{n\pi}{L}xdx\,.$$

*2.1 Problem Solving*

In order to transform Eq. (1) into homogeneous, we substitute $T - T_a = u$ in Eq. (1), so that Eq. (15) is obtained with its conditions (16) and (17).

$$D\frac{\partial^2 u}{\partial x^2} - Hu = \frac{\partial u}{\partial t} \ \ ,0 < x < L\,, t > 0 \tag{15}$$

$$u(0,t) = T_A - T_a,\ u(L,t) = T_B - T_a \tag{16}$$

$$u(x,0) \ = f(x) - T_a \tag{17}$$

Now it is replaced $u(x,t) = V(x,t) + \psi(x)$ in Eq. (15), obtaining:

$$D\frac{\partial^2 V}{\partial x^2} + D\frac{d^2\psi}{dx^2} - HV - H\psi = \frac{\partial V}{\partial t}$$

This equation can be separated, and equalize the separation constant to zero, in order that the equations are homogeneous, which will allow to use the method of separation of variables

$$D\frac{\partial^2 V}{\partial x^2} + -HV - \frac{\partial V}{\partial t} = -D\frac{d^2\psi}{dx^2} + H\psi = 0$$

From this assumption, two differential equations to be solved are derived, Eqs. (18) and (19); when it is applying the conditions (16) and (17) to this equations, two non-homogeneous boundary conditions are obtained, as can be seen in the Eqs. (20)-(22).

$$-D\frac{d^2\psi}{dx^2} + H\psi = 0 \tag{18}$$

$$D\frac{\partial^2 V}{\partial x^2} - HV - \frac{\partial V}{\partial t} = 0 \tag{19}$$

$$V(0,t) = T_A - T_a - \psi(0) \tag{20}$$

$$V(L,t) = T_B - T_a - \psi(L) \tag{21}$$

$$V(x,0) = f(x) - \psi(x) - T_a \tag{22}$$

From Eqs. (20) and (21) it follows that $V(0,t)$ and $V(0,t)$ are constants, which are chosen as $V(0,t) = V(\mathrm{L},t) = 0$ to have conditions of homogeneous boundary and thus be able to use the method of separation of variables in Eq. (19). The boundary conditions and the initial condition that will be applied in the resolution of Eqs. (18) and (19) are shown in (23).

$$\begin{cases} \psi(0) = T_A - T_a \\ \psi(L) = T_B - T_a \\ V(x,0) = f(x) - \psi(x) - T_a \end{cases} \tag{23}$$

The general solution of the ordinary differential Eq. (18) is the Eq. (24):

$$\psi(x) = c_1 cosh\left(\sqrt{H/D}\,x\right) + c_2 sinh\left(\sqrt{H/D}\,x\right) \tag{24}$$

where $c_1$ and $c_2$ are obtained by applying the boundary conditions in (23), result:

$$c_1 = T_A - T_a\ , \qquad c_2 = \frac{T_B - T_a - (T_A - T_a)\, cosh\left(\sqrt{H/D}\,L\right)}{sinh\left(\sqrt{H/D}\,L\right)}.$$

To solve Eq. (15) we substitute the expression the product of functions $V(x,t) = \phi(x)\,h(t)$, obtaining $\phi''/\phi = (h' + \mathrm{H}h)/Dh$, since these expressions must be equal to a constant, the constant$-\lambda^2$ is chosen which guarantees a solution with physical significance for both ordinary differential equations, $\phi''/\phi = (h' + \mathrm{H}h)/Dh = -\lambda^2$, from which the differential equations are obtained

$$\phi'' + \lambda^2\phi = 0 \tag{25}$$

$$h' + (H + D\lambda^2)h = 0 \tag{26}$$

whose solutions are:

$$\phi(x) = c_3\, cos(\lambda x) + c_4\, sin(\lambda x) \tag{27}$$

$$h(t) = c_5 e^{-(H+D\lambda^2)t} \tag{28}$$

Substituting Eqs. (27) and (28) in $V(x,t) = \phi(x)\, h(t)$ we obtain the preliminary solution presented in the Eq. (29), where $c_3 = 0;\ c_4 \sin(\lambda L) = 0 \rightarrow \lambda L = n\pi \rightarrow \lambda = \frac{n\pi}{L},\ n = 1,2,3,4 \ldots.;\ A = c_4 c_5$, this is due to the boundary conditions $V(0,t) = V(L,t) = 0$.

$$V(x,t) = [c_3\, cos(\lambda x) + c_4\, sin(\lambda x)]\left[c_5 e^{-(H+D\lambda^2)t}\right] = Ae^{-(H+D\lambda^2)t} sin\left(\frac{n\pi}{L}x\right) \tag{29}$$

Now $A$ must be determined, for which the initial condition $V(x,0) = f(x) - \psi(x) - T_a$ is applied, in this case, we cannot find a simple value for the coefficient $A$ that satisfies With this condition, this coefficient is obtained by applying the orthogonality of the sine function on the interval $(0, L)$. By means of the superposition principle [15] the solution for Eq. (19) is established:

$$V(x,t) = \sum_{n=1}^{\infty} A_n e^{-(H+D\lambda^2)t} sin\left(\frac{n\pi}{L}x\right) \tag{30}$$

where

$$A_n = \frac{2}{L}\int_0^L [f(x) - \psi(x) - T_0]\, sin\left(\frac{n\pi}{L}x\right) dx$$

Finally, we have that the solution of Eq. (1), subject to the boundary conditions (2) and initial condition (3) is $T(x,t) = V(x,t) + \psi(x) + T_a$, that is to say:

$$T(x,t) = e^{-Ht}\sum_{n=1}^{\infty} A_n e^{-D\lambda^2 t} sin\left(\frac{n\pi}{L}x\right) + c_1 cosh\left(\sqrt{H/D}\, x\right) + c_2 sinh\left(\sqrt{H/D}\, x\right) + T_a \tag{31}$$

where:

$$c_1 = T_A - T_a$$
$$c_2 = \frac{T_B - T_a - (T_A - T_a)cosh\left(\sqrt{H/D}\, L\right)}{sinh\left(\sqrt{H/D}\, L\right)}$$
$$A_n = \frac{2}{L}\int_0^L [f(x) - \psi(x) - T_a]\, sin\frac{n\pi}{L}x dx,\ n = 1,2,3,4, \ldots..$$

*2.2 Deduction of Eq. (1)*

Let us consider any segment of finite length , $x = a$ to $x = b$, of the rod (see Figure 2), in this figure $\phi(x,t)$ is the heat flux, quantity of thermal energy per unit time that flows through cross section A per unit area, in J/m2s ; $Q(x,t)$ is the thermal energy generated per unit volume and per unit time, in J/m3s.

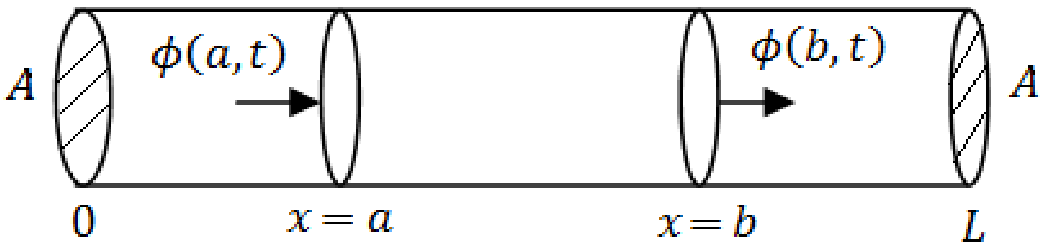


**Fig. 2.** Heat flux in a through a cross section A in an interval of length $[a, b]$

The total thermal energy in the interval $[a, b]$ is $E_{T,total} = \int_a^b e(x,t)A\,dx$, where $e(x,t)$ is the thermal energy density in J/m$^3$ and A is transverse of the rod in $\mathrm{m}^2$, as shown in Eq. (32). For conservation of energy, this energy must equal the energy transferred on the cross surface ($E_{Cross}$) and the energy transferred on the lateral surfaces ($E_{Lateral}$), as well as the energy due to heat sources, as shown in Eq. (35). In Eq. (34), $w(x,t)$ is a surface energy density in J/m$^2$, defined for the energy transferred on the lateral surface.

$$E_{T,total} = \int_a^b e(x,t)A\,dx = E_{Cross} + E_{Lateral} + E_Q \tag{32}$$

$$E_{Cross} = \int_S \phi(a,t)dA + \left(-\int_S \phi(b,t)dA\right) = A[\phi(a,t) - \phi(b,t)] = -A\int_a^b \frac{\partial\phi}{\partial x}dx \tag{33}$$

$$E_{Lateral} = \int_{Surface} w(x,t)dA = \int_{x=a}^{x=b}\int_{\theta=0}^{\theta=2\pi} w(x,t)\,dx\,rd\theta = 2\pi r\int_a^b w(x,t) \tag{34}$$

$$E_Q = \int_a^b Q\,Adx = A\int_a^b Q\,dx \tag{35}$$

Substituting these last three expressions in the Eq. (32) we obtain

$$\int_a^b \frac{\partial e}{\partial t}\,dx = -A\int_a^b \frac{\partial\phi}{\partial x}dx + 2\pi r\int_a^b w(x,t)\,dx + A\int_a^b Q\,dx$$
$$\int_a^b \left(\frac{\partial e}{\partial t} + \frac{\partial\phi}{\partial x} - Q - 2\pi r\frac{w}{A}\right)\,dx = 0$$

where from

$$\frac{\partial e}{\partial t} = -\frac{\partial\phi}{\partial x} + Q + 2\pi r\frac{w}{A} \tag{36}$$

Specific heat and temperature can be related by the equation ecuación $e(x,t) = \rho\,c(x)\,T(x,t)$ [8], substituting this expression in Eq. (36), the Eq. (37) is obtained.

$$c(x)\rho(x)\frac{\partial T}{\partial t} = -\frac{\partial\phi}{\partial x} + Q + 2\pi r\frac{w}{A} \tag{37}$$

The next step is to apply Fourier's law of heat conduction, which is based on the following observations: 1) if the temperature is constant in a region, the thermal energy does not flow; 2) If there are temperature differences, the energy flows from the hottest to the coldest region; 3) the greater the temperature difference (for the same material), the greater the flow of thermal energy; 4) the flow of thermal energy is different for different materials, even with the same temperature difference. These observations are summarized by Eq. (38), known as Fourier's law. In Eq. (38), $\partial T/\partial x$

represents the change in temperature per unit length, and the sign "-" indicates that the heat flow is in the direction in which the temperature decreases. The coefficient of proportionality $\mathrm{K}_0$ measures the ability of the material to conduct heat and is called thermal conductivity. Experiments indicate that $\mathrm{K}_0$ depends on the type of material. A material with a high value of $\mathrm{K}_0$ is a good conductor of thermal energy, whereas a material with a small value of $\mathrm{K}_0$ could function as an insulator of thermal energy.

$$\phi = -K_0 \frac{\partial T}{\partial x} \tag{38}$$

Substituting $\phi$ from Eq. (38) into Eq. (37) we obtain Eq. (39).

$$c\rho \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial T}{\partial x}\right) + Q + \frac{2\pi r}{A} w \tag{39}$$

To continue with the deduction, the following is assumed:

i) There are no sources, that is, $Q = 0$
ii) The rod is uniform, so $c$ , $\rho$ , $y$ $K_0$ are constants
iii) The speed with which the energy (power) flows in the lateral area is given by Newton's law of cooling: $P_l = -hA_l(T - T_0)$ , where $h$ is the convection coefficient (the larger this value, faster energy is transferred) The units of $h$ are $\mathrm{W/m^2K}$.

Since the intensity in the lateral area (power per unit area) is represented, it will be expressed as

$$w = \frac{P_l}{A_l} = -h(T - T_0) \tag{40}$$

Applying these assumptions in Eq. (39), the Eq. (1) is obtained:

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2} - H(T - T_0).$$

In the Eq. (1), $H$ is the convective thermal conductance with units of $\mathrm{s}^{-1}$ and $D$ is the thermal diffusivity of the rod with units of $\mathrm{m}^2/\mathrm{s}$ in the SI units' system. The expressions to obtain the values of $H$ and $D$ from more basic properties are shown in Eqs. (41) and (42), respectively.

$$H = \frac{2\pi r}{c\rho A} h = \frac{2\pi r}{c\rho\pi r^2} h = \frac{2h}{c\rho r} \tag{41}$$

$$D = \frac{K_0}{c\rho} \tag{42}$$

## 3. Discussion

### *3.1 Analytic Expression Analysis*

It can be easily verified that Eq. (31) satisfies the boundary conditions (2) and (3). Regarding the initial condition, for an infinite summation of terms in $V(x,t)$ with $t = 0$, the summation must converge to the function $f(x) - \psi(x) - T_a$ that when added to $\psi(x) + T_a$ would give $f(x)$ which is what is required in condition (3).

When we take $T(0,t) = T_A = T_a$ , $T(L,t) = T_B = 0, L = 1$ in Eq. (31), we obtain Eq. (43), which is the general solution of Eq. (4) with its boundary conditions in the Eqs. (5) and (6). Taking the limit $\lim_{t\to 0} T(x,t)$ in (43) we obtain the solution in steady state of (4). Therefore, function (43) is a more general solution than Eq. (30).

$$T(x,t) = e^{-Ht} \sum_{n=1}^{\infty} A_n e^{-D\lambda^2 t} sin\left(\frac{n\pi}{L}x\right) + T_a\left(1 - \frac{sinh(\sqrt{H/D}x)}{sinh(\sqrt{H/D})}\right) \tag{43}$$

Solution (31) cannot be considered as a solution that models the case $H = 0$ directly, since this solution becomes indeterminate when H equals zero, because $c_2$ tends to infinity when $H$ tends to zero. Next, it is determined whether solution (31) converges to Eq. (14) when $H$ approaches zero.

We take the limit $H \to 0$ in Eq. (31),

$$\lim_{H\to 0} T(x,t) = \lim_{H\to 0}\left\{e^{-Ht} \sum_{n=1}^{\infty} A_n e^{-D\lambda^2 t} sin\left(\frac{n\pi}{L}x\right)\right\} + \lim_{H\to 0}\{\psi(x)\} + T_a$$
$$= \sum_{n=1}^{\infty} A_n e^{-D\lambda^2 t} sin\left(\frac{n\pi}{L}x\right) + \lim_{H\to 0}\{\psi(x)\} + T_a$$

It is observed that the time-dependent part of (31) converges to the time-dependent part in Eq. (14), $\lim_{H\to 0} V(x,t) \approx v_1(x,t)$, now we work with the independent part of time $\lim_{H\to 0}\{\psi(x)\} + T_a$.

$$\lim_{H\to 0}\{\psi(x)\} + T_a \approx \lim_{H\to 0}\left\{c_1 cosh\left(\sqrt{H/D}\,x\right) + c_2 sinh\left(\sqrt{H/D}\,x\right)\right\} + T_a$$
$$= \lim_{H\to 0}\left\{c_1 cosh\left(\sqrt{H/D}\,x\right)\right\} + \lim_{H\to 0}\left\{c_2 sinh\left(\sqrt{H/D}\,x\right)\right\} + T_a$$
$$= T_A - T_a + \lim_{H\to 0}\left\{c_2 sinh\left(\sqrt{H/D}\,x\right)\right\} + T_a$$
$$= \lim_{H\to 0}\left\{\frac{T_B - T_a - (T_A - T_a)cosh(\sqrt{H/D}\,L)}{sinh(\sqrt{H/D}\,L)} senh\left(\sqrt{H/D}\,x\right)\right\} + T_A$$
$$= \lim_{H\to 0}\left\{T_B - T_a - (T_A - T_a)cosh\left(\sqrt{H/D}\,L\right)\right\} \times \lim_{H\to 0}\left\{\frac{sinh(\sqrt{H/D}\,x)}{sinh(\sqrt{H/D}\,L)}\right\} + T_A$$
$$= T_A + (T_B - T_A) \times \lim_{H\to 0}\left\{\frac{sinh(\sqrt{H/D}\,x)}{sinh(\sqrt{H/D}\,L)}\right\}$$

It is observed in this last expression, that when substituting $H = 0$ the expression $\sinh\left(\sqrt{H/D}x\right)/\sinh\left(\sqrt{H/D}\,L\right)$ gives an indeterminacy 0/0, this indeterminacy allows to use the rule of l'Hôpital:

$$\lim_{H\to 0}\{\psi(x)\} + T_a \approx (T_B - T_A) \times \lim_{H\to 0}\left\{\frac{\frac{\partial}{\partial H} sinh(\sqrt{H/D}\,x)}{\frac{\partial}{\partial H} sinh(\sqrt{H/D}\,L)}\right\} + T_A$$
$$= (T_B - T_A) \times \lim_{H\to 0}\left\{\frac{\frac{1}{2D}\left(\frac{H}{D}\right)^{-\frac{1}{2}} x\, cosh\left(\sqrt{\frac{H}{D}}x\right)}{\frac{1}{2D}\left(\frac{H}{D}\right)^{-\frac{1}{2}} L\, cosh\left(\sqrt{\frac{H}{D}}L\right)}\right\} + T_A$$
$$= (T_B - T_A) \times \frac{x \lim_{H\to 0}\{cosh(\sqrt{H/D}\,x)\}}{L \lim_{H\to 0}\{cosh(\sqrt{H/D}\,L)\}} + T_A$$
$$= (T_B - T_A) \times \frac{x}{L} + T_A = \left(\frac{T_B - T_A}{L}\right)x + T_A\,.$$

It has then been confirmed that $\lim_{H \to 0} T(x,t) \approx T_1(x,t)$, see in the Eq. (14). This result implies that the solution in the Eq. (43) can be used for the case $H = 0$, approaching $H$ to zero, in the next section a numerical evaluation is carried out to show this approximation.

### *3.2 Numerical Evaluation of the Analytical Solution*

To numerically evaluate the analytical solution, use the data in Table 1, and the calculated parameters from Table 2. The values of $D$ and $H$ are determined with Eqs. (41) and (42). The temperature distribution in the rod is shown will assume as constant to simplify the calculations in this case $f(x) = 15.0^{\circ}$C. The evaluation is carried out in MATLAB [16], 200 terms are used in the Fourier series, the solution (22) is evaluated in 8 times, from zero seconds to 5400 s (90 min). Values of $x$ range from $x = 0$ to $x = 0.500\text{m}$ in steps of 0.0025 m. With these data, the graphs in Figure 3 and Figure 4 are constructed, which show the temperature distribution along the rod, for $H = 8.3043 \times 10^{-5} s^{-1}$ and $H = 8.3043 \times 10^{-25} s^{-1}$, respectively.

Figure 3 shows that $x = 0$ and $x = 0.500\text{m}$, the solution (30) diverges considerably from the expected value $f(x) = 15.0^{\circ}$C, this is because these are points of discontinuity, for which the Gibbs phenomenon is presented. When 90 minutes have passed, a distribution is shown according to the function $\psi(x) = c_1 \cosh\left(\sqrt{H/D}\, x\right) + c_2 \sinh\left(\sqrt{H/D}\, x\right)$ that has the shape of a concave curve towards above. It is observed in Figure 4, that when $H \approx 0$ the final temperature distribution converges to the linear function $\psi_1(x)$ as it was demonstrated analytically.

**Table 1**
Rod parameters and other data in equation (1)

| Parameter | Value | Units |
|---|---|---|
| Material: steel | -- | -- |
| Density ($\rho$) | 7850 | kg/m$^3$ |
| Thermal conductivity ($K_0$) | 50 | W/ms |
| Specific heat ($c$) | 460 | J/kg•$^{\circ}$C |
| Length ($L$) | 0.500 | m |
| Diameter ($d$) | 0.0800 | m |
| Convection coefficient[1] ($h$) | 5.867 | W/m$^{2\circ}$C |
| Temperature at the extreme A ($T_A$) | 70.0 | $^{\circ}$C |
| Temperature at the extreme B ($T_B$) | 20.0 | $^{\circ}$C |
| Temperature $T_a$ | 25.0 | $^{\circ}$C |

**Table 2**
Calculated parameters

| Parameter | Formula | Value | Units |
|---|---|---|---|
| Side area | $A_l = (\pi d)L$ | 0.1257 | m$^2$ |
| Diffusivity | $D = K_0/(\rho c)$ | $1.41\text{x}10^{-5}$ | m$^2$/s |
| convective thermal conductance | $H = 2h/(c\rho r)$ | $8.30\text{x}10^{-5}$ | s$^{-1}$ |

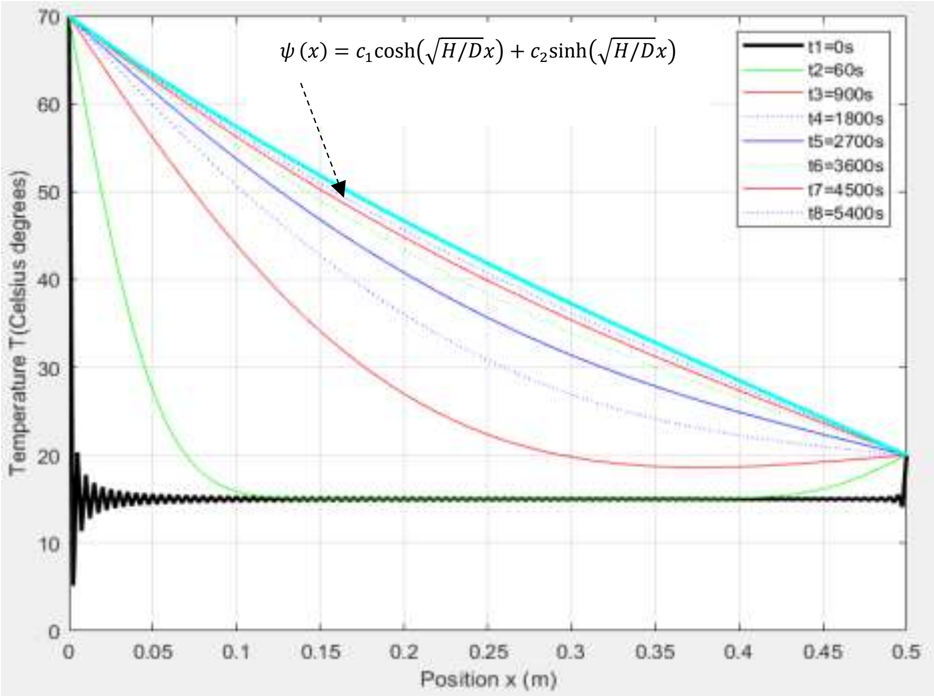


**Fig. 3.** Temperature distribution for $H = 8.3043 \times 10^{-5} s^{-1}$

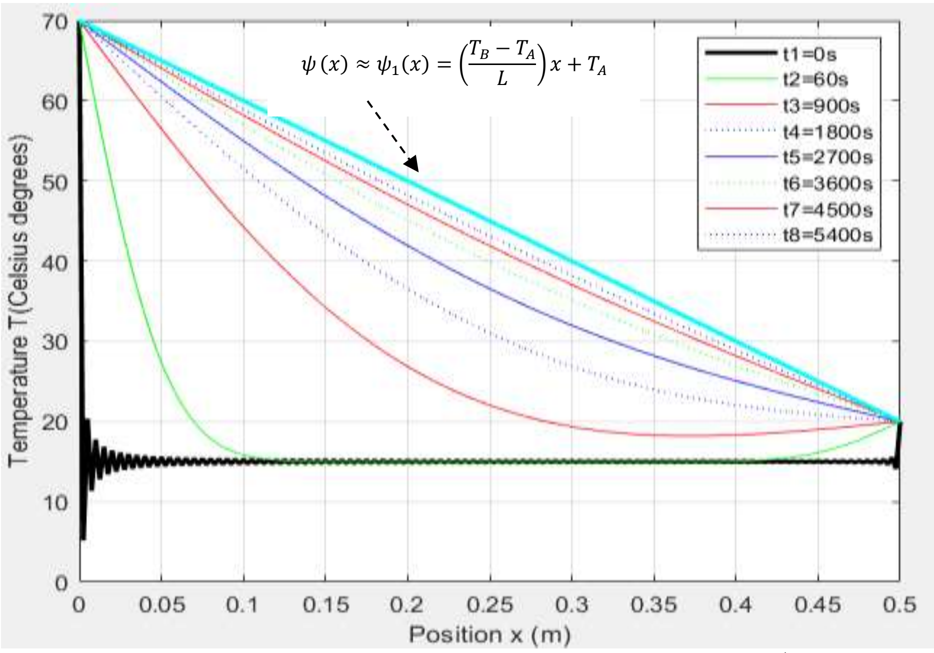


**Fig. 4.** Temperature distribution for $H = 8.3043 \times 10^{-25} s^{-1}$

## 4. Conclusions

In this article the problem of heat conduction along a rod with heat transfer by convection in the lateral surface was approached, the boundary conditions were those of Dirichlet and the condition the initial temperature distribution was an arbitrary function of $x$. The development of this problem leads to the non-homogeneous one-dimensional heat equation with boundary conditions that are also non-homogeneous. This equation was solved using the substitutions $u(x,t) = T(x,t) - T_a$ and $u(x,t) = V(x,t) + \psi(x)$, and imposing homogeneous boundary conditions in the derived equations of the procedure. The obtained function was evaluated analytically and also numerically in MATLAB to analyze the temperature distribution in the rod. The main conclusions are presented below:

i) The solution found for the one-dimensional heat equation with convection heat transfer on the lateral surface, meets the Dirichlet boundary conditions. In the case of the initial condition, the convergence is not good at the ends of the rod due to the Gibbs phenomenon in the Fourier series.

ii) The solution obtained presents a discontinuity when the convective thermal conductance is zero, $H = 0$. However, when taking the limit $H \to 0$ in the solution obtained, it converges to the solution of the case in which there is no convection $H = 0$, so that the solution obtained can be considered as a general solution, since it can model convection and non-convection on the lateral surface.

iii) The steady-state solution of the solution to the heat equation raised (1) is a linear combination of the functions cosh and sinh, which presents a slight concavity, while if convection were negligible, the steady-state distribution it would be linear.

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