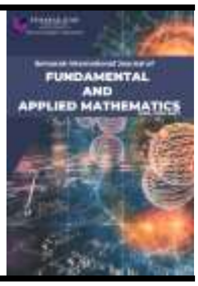
Semarak International Journal of Fundamental and Applied Mathematics

Journal homepage:
https://semarakilmu.my/index.php/sijfam
ISSN: 3030-5527



An Efficient Semi-Analytical Method by Using Adaptive Approach in Solving Nonlinear Schrödinger Equations

Abdul Rahman Farhan Sabdin[1], Che Haziqah Che Hussin[2,*], Arif Mandangan[1], Jumat Sulaiman[1]

[1] Faculty of Science and Natural Resources, Universiti Malaysia Sabah, Jalan UMS, 88400 Kota Kinabalu, Sabah, Malaysia
[2] Preparatory Centre of Science and Technology, Universiti Malaysia Sabah, Jalan UMS, 88400, Kota Kinabalu, Sabah, Malaysia

ARTICLE INFO

Article history:
Received 13 January 2024
Received in revised form 15 February 2024
Accepted 18 March 2024
Available online 30 June 2024

Keywords:
Adaptive multistep differential transform method; multistep reduced differential transform method; nonlinear schrodinger equations; adaptive algorithm; adomian polynomials

ABSTRACT

This paper introduces a novel method called the Adaptive Hybrid Reduced Differential Transform Method (AHRDTM) to solve Nonlinear Schrödinger Equations (NLSEs). This method provides semi-analytical solutions over a longer time frame. It achieves this by producing sub-division intervals of varying lengths, distinguishing it from the classical Multistep Reduced Differential Transform Method (MsRDTM). Importantly, the AHRDTM eliminates the necessity for perturbation, linearization, or discretization, providing the benefits of adaptability and reliability. The outcomes exhibit that AHRDTM yields highly efficient solutions for NLSEs. Moreover, the method is simple, significantly reducing the computational workload in solving NLSE problems, and shows promising opportunity for application in diverse complex partial differential equations (PDEs). The efficiency and effectiveness of AHRDTM are demonstrated through tables and graphical representations.

1. Introduction

Partial differential equation (PDE) is an important form of differential equations (DEs) to explain and model scientific phenomena in optics, acoustics, fluid mechanics, hydrodynamics, and astronomy [1]. This concept and its application have always been important in the research areas of pure and modern mathematics [2]. Due to highly complex problems in solving PDEs, numerous methods have been introduced to solve such problems. A few analytical techniques used such as the Fourier Spectral Method [3], and accelerated Adomian Decomposition Method (ADM) [4] to solve these applied models since their structural complexity is typically high. Some effectual methods in solving PDEs are the Symmetry Reduction Strategy [5], and Hirota Bilinear Forms [6].

Several hyperbolic wave-type equations have been used to serve as mathematical models for waves such as Korteweg-De Vries Equations (KdVEs), Klein-Gordon Equations (KGEs), Harry Dym Equations, Burgers Equations, and Schrödinger Equations (SEs). In this paper, the Nonlinear SEs

* *Corresponding author.*
E-mail address: haziqah@ums.edu.my

https://doi.org/10.37934/sijfam.2.1.112b

(NLSEs) which is one of the most famous equations, is a mathematical model that appears in various fields, including hydrodynamics, plasma waves, nonlinear optical waves, oceanography, deep surface water waves, ocean roughness, biology, quantum mechanics, and light emission in cables of fibre optics [7]. This equation has been evaluated both numerically and analytically employing diverse methods, including the Modified Generalized Exponential Rational Function Method (mGERFM) [8], the Modified Exponential Jacobi Method [9], and several other methods [10,11]. Approximate analytical solutions which also known as semi-analytical methods are also useful in solving NLSEs, such as the Shehu Transform based Adomian Decomposition Method (STADM). It has been used to solve time fractional NLSEs [12]. Then, Jawad *et al.,* [13] applied the Variational Homotopy Transform Method (VHTM) to solve SEs. An important analytic method is the Differential Transform Method (DTM). The provided DE in this method is accompanied by the initial conditions, which are transformed into a recursive equation, ultimately resulting in the solution represented as a Taylor series. DTM is also the original method used in this paper.

DTM and its advanced variants have been brought into light by scientists. Initially, the Reduced DTM (RDTM) have been introduced in 2009 by Keskin *et al.,* [14] to tackle PDEs problems. This semi-analytical method gained plenty attention since it helps in solving a variety of problems by many researchers [15]. However, there are drawbacks when handling complex PDEs. Due to the complexity in solving fractional KdVEs, Ray [16] suggested an alteration to the fractional RDTM. This modification involves replacing the nonlinear term with Adomian polynomials, making it possible to derive solutions more easily and with fewer calculated terms. Thus, this modification of the RDTM is a great approach in dealing problems with nonlinear terms of high nonlinearities. The method is called Modified RDTM (MRDTM).

Another improvement of the DTM is the Multistep DTM (MsDTM). This semi-analytic method was first introduced in 2010 by Odibat *et al.,* [17] and applied to various systems. It produces a solution where its convergent series rapidly converges in a large time frame over a sequence of equal-length subintervals which then improves the convergence of the series solution. Other researchers, Al-Smadi *et al.,* and Momani *et al.,* also used the multistep scheme by applying it on RDTM, called the Multistep RDTM (MsRDTM) to solve fractional PDEs [18]. A combination of the MsRDTM and the MRDTM was then introduced in 2018 and 2019. Che Hussin *et al.,* [19] proposed and implemented the Multistep Modified RDTM (MMRDTM) in solving NLSEs. This method has also been used by Sabdin *et al.,* [20,21] in solving Nonlinear Telegraph Equations (NLTEs), and time fractional NLTEs (TFNLTEs) with source term.

However, the main drawback of using a multistep scheme is that it is less efficient when considering large intervals. Thus, an adaptive approach is essential to solve such problems. This is important to enable the utilization of variable-length step-sizes. Consequently, the problems solved will converge over a large time frame over a sequence of subintervals with varied length resulting in fewer numbers of time-steps. Two distinct adaptive algorithms with different approaches have been developed by Gokdogon *et al.,* [22] and El-Zahar [23]. Including the references mentioned before, both adaptive schemes on DTM called the Adaptive MsDTM (AMsDTM) have been implemented to obtain approximate analytical solutions for nonlinear problems [24].

This paper aims to implement a new efficient and novel approach named the Adaptive Hybrid RDTM (AHRDTM) to solve NLSEs. The term adaptive hybrid indicates the combination of the adaptive multistep approach of AMsDTM by El-Zahar [23] and the modification of RDTM (MRDTM) by using suitable Adomian polynomials. The adaptive algorithm has been implemented to get solutions with variable length subintervals. The tables provided show the comparison of number of time-steps, and the processing time between the classical Modified MsRDTM (MMsRDTM), and the proposed method, AHRDTM. The finding indicates the efficiency of the new efficient approach, AHRDTM in solving the considered equations. The remaining portions of this work are organized as follows. Section 2 discusses the definitions, adaptive algorithm, and solution formulations. Section 3 illustrates the application of the AHRDTM in several NLSEs with solutions presented in tables and graphical illustrations. Lastly, Section 4 provides concluding observations.

2. Development of Adaptive Hybrid Reduced Differential Transform Method

The development of this novel method consists of methods from Reduced Differential Transform Method (RDTM), Modified Reduced Differential Transform Method (MRDTM), Multistep Reduced Differential Transform Method (MsRDTM), and Adaptive Multistep Differential Transform Method (AMsDTM). The steps, and disadvantages related to the methods are mentioned below.

2.1 Reduced Differential Transform Method

Two-variable function $u(x,t)$ is considered that may be written as the product of two single-variable functions: $u(x,t) = f(x)g(t)$. On the foundational properties of the one-dimensional differential transform, the function $u(x,t)$ may be written as follows:

$$u(x,t) = \left(\sum_{i=0}^{\infty} F(i)x^i\right)\left(\sum_{j=0}^{\infty} G(j)t^j\right) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{1}$$

where the t-dimensional span function of $u(x,t)$ is denoted by $U_k(x)$. The following [14] are the fundamental definitions of RDTM:

Definition 1. If the domain of interest's function $u(x,t)$ is analytical and continuously differentiable with regard to time t and space x, then letting

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0}, \tag{2}$$

where the transformed function is the t-dimension span function $U_k(x)$. In this paper, the primary function is denoted by the small letter $u(x,t)$, while the altered function is symbolized by the capital letter $U_k(x)$.

Definition 2. Given the following for the differential inverse transform of $U_k(x)$:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{3}$$

Then, by fusing (2) and (3), we obtain

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0} t^k. \tag{4}$$

According to the preceding definitions, the RDTM concept is obtained from the expanded power series. Consider the following operator-form nonlinear PDE to explain the fundamental RDTM concepts as in (5)

$$\mathcal{L}u(x,t) + \mathcal{R}u(x,t) + \mathcal{N}u(x,t) = g(x,t), \tag{5}$$

with initial condition
$u(x,0) = f(x),$ (6)

where $\mathcal{L} = \frac{\partial}{\partial t}$, $\mathcal{R}$ is a partial derivatives linear operator, $\mathcal{N}u(x,t)$ is a nonlinear operator and $g(x,t)$ is an inhomogeneous term.

Based on RDTM, the iteration formula shown below may be formed:

$$(k + 1)U_{k+1}(x) = \mathcal{G}_k(x) - \mathcal{R}U_k(x) - \mathcal{N}U_k(x), \tag{7}$$

where $U_k(x), \mathcal{R}U_k(x), \mathcal{N}U_k(x)\ and\ \mathcal{G}_k(x)$ are the transformations of the functions $\mathcal{L}u(x,t), \mathcal{R}u(x,t),\ \mathcal{N}u(x,t), and\ g(x,t)$ respectively. Based on initial condition (6), we write

$$U_0(x) = f(x). \tag{8}$$

2.2 Modified Reduced Differential Transform Method

The nonlinear term in equation (7) is denoted by Ray [16] as follows:

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n\big(U_0(x), U_1(x), \dots, U_n(x)\big). \tag{9}$$

The method proposed for computing the Adomian polynomials as shown

$$A_0 = N\big(U_0(x)\big), \tag{10}$$

$$A_n(U_0(x), U_1(x), \dots, U_n(x)) = \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{n} \lambda^k U_k(x)\right)\right]\right)_{\lambda=0}, n \geq 1, \tag{11}$$

such that $\mathcal{N}U_k(x)$ is the term of nonlinearity. Substituting the nonlinear term by its Adomian polynomial yields,

$$(k + 1)U_{k+1}(x) = \mathcal{G}_k(x) - \mathcal{R}U_k(x) - A_k. \tag{12}$$

Notice that the approach does not require time-consuming computations with high derivatives. Iterative calculation can be used to obtain $U_k(x)$ values by combining equations (8) into (12). Furthermore, the set of inverse transformation values, $\{U_k(x)\}_{k=0}^{n}$ yields the approximate solution as follows:

$$u_n(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k, t \in [0,T]. \tag{13}$$

2.3 Multistep Reduced Differential Transform Method

The MsRDTM identifies RDTM series by dividing the interval $[0,T]$ into R subintervals of equal length. Nonetheless, addressing linear and nonlinear DEs necessitates opting for a small-time step size, which, in turn, involves obtaining RDTM solutions across more subintervals. Consequently, selecting a smaller time step size and increasing the number of subintervals result in longer computation times. To address this challenge, there is a need for a novel approach, and an adaptive methodology is suggested as a solution.

2.4 Adaptive Multistep Differential Transform Method

The following adaptive time step-size control algorithm is taken from the algorithm introduced in AMsDTM. The algorithm is then applied according to the adaptive scheme by [23]:

(a) One specifies the acceptable local error $\delta > 0$, and selects the order N of the multistep scheme.
(b) Based on computations, the values $\left|U_{k,r}(N)\right|$, $k = 1, 2, \ldots, n$, are determined for each solution component k.
(c) At the grid point t_r, we compute the value $E_N = max\left(\left|U_{k,r}(N)\right|\right)$, $k = 1, 2, \ldots, n$.
(d) We choose the step-size h_r such that $h_r = \tau \left(\frac{\delta}{E_N}\right)^{\frac{1}{N}} \leq h_{max}$ and $t_{r+1} = t_r + h_r$, where τ is a safety factor, and h_{max} is the maximum allowed step-size.

Firstly, MRDTM is applied to the initial value problem of interval $[0, t_1]$. Then, by using the initial conditions

$$u(x, 0) = f_0(x). \tag{14}$$

We obtain the approximate result

$$u_1(x, t) = \sum_{k=0}^{\mathrm{k}} U_{k,1}(x) t^k, t \in [0, t_1]. \tag{15}$$

The adaptive step-size control algorithm is then applied to determine t_1 of $[0, t_1]$.
Then, for each subinterval $[t_{r-1}, t_r]$, the initial condition

$$u_r(x, t_{r-1}) = u_{r-1}(x, t_{r-1}), \tag{16}$$

is used for $r \geq 2$ and the implementation of AHRDTM to the initial value problem on $[t_{r-1}, t_r]$. For $r = 1,2, \ldots , R$, the algorithm is then applied repeatedly for each subinterval $[t_{r-1}, t_r]$ of R variable-length subintervals to determine each t_r. Thus, the interval $[0, T]$ is a combination of variable-length subintervals, $[t_{r-1}, t_r]$ for $t \epsilon [0, T]$.

The repetition of the process is performed and carried out to construct an approximate solutions sequence $u_r(x, t)$ for $r = 1,2, \ldots, R$, such as,

$$u_r(x, t) = \sum_{k=0}^{K} U_{k,r}(x)(t - t_{r-1})^k, t \in [t_{r-1}, t_r]. \tag{17}$$

Finally, the AHRDTM proposes the following solutions:

$$u(x, t) = \begin{cases} u_1(x, t), & for\ t \in [0, t_1] \\ u_2(x, t), & for\ t \in, [t_1, t_2] \\ \vdots \\ u_R(x, t), & for\ t \in, [t_{R-1}, t_R] \end{cases}. \tag{18}$$

It is crucial to note that when the step size $s = T$, the MRDTM is derived from AHRDTM.

3. Results

Two numerical examples have been considered to show the reliability of the AHRDTM as its benefit for solving NLSEs:

Example 1. Cubic NLSE of the form [25]

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{19}$$

is considered with initial condition

$$u(x,0) = e^{ix}. \tag{20}$$

$e^{i(x+t)}$ is the exact solution of this equation.

By applying the AHRDTM to equation (19) and using fundamental properties of AHRDTM, we have

$$U_{k+1,\mathrm{r}}(x) = \left(\frac{I}{k+1}\right)\left(\frac{\partial^2}{\partial x^2} U_{k,r}(x) + 2A_{k,r}\right). \tag{21}$$

We write the transformed initial condition (20) as

$$U_0(x) = e^{ix}. \tag{22}$$

The adaptive algorithm is then implemented to obtain an approximate solution.

The performance analysis is summarized in Table 1, Table 2, Table 3, and Table 4. The tables present comparisons of the processing time and the time-step, R used in solving (19) by the MMsRDTM, and AHRDTM at N = 6, with respect to the specified tolerance, δ over the interval $[0,5]$ for different values of h.

Table 1, Table 2, Table 3, and Table 4 show that AHRDTM is significantly more efficient than the classical MMsRDTM by observing the processing time and number of time-steps needed in approximating solutions for this problem. Thus, through observations of the processing time with different values of h for each table, the presented adaptive approach, AHRDTM demonstrates notable speed and efficiency.

Table 1
The duration of processing time and the total numbers of time-step for $\tau = 1.0$, $h = 0.1$, $t \in [0,5]$

MMsRDTM		AHRDTM		
Processing time (s)	Time-step, R	Admissible local error, δ	Processing time (s)	Time-step, R
		0.1	0.828	3
12.592	50	0.01	1.048	4
		0.001	1.640	6

Table 2
The duration of processing time and the total numbers of time-step for $\tau = 1.0$, $h = 0.01$, $t \in [0, 5]$

MMsRDTM		AHRDTM		
Processing time (s)	Time-step, R	Admissible local error, δ	Processing time (s)	Time-step, R
		0.1	0.845	3
138.400	500	0.01	1.108	4
		0.001	1.656	6

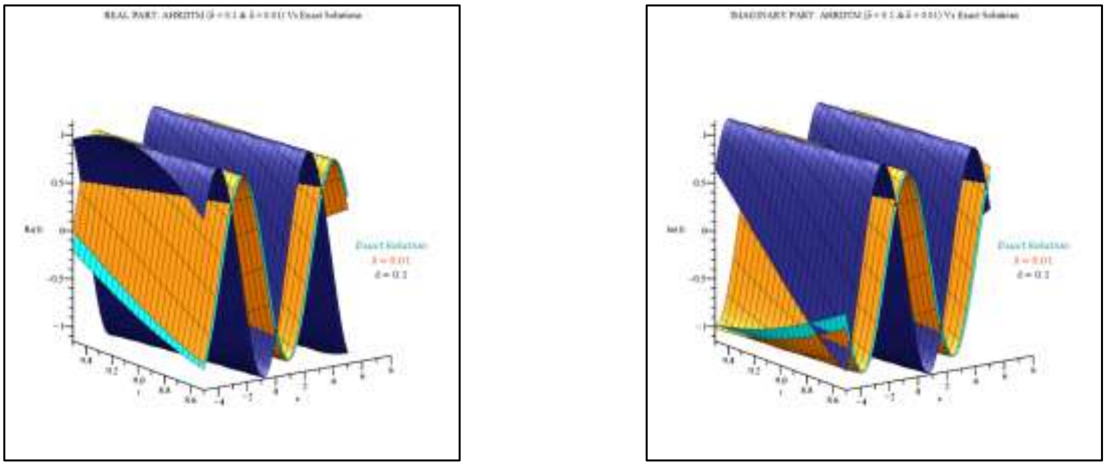
Table 3
The duration of processing time and the total numbers of time-step for $\tau = 1.0$, $h = 0.001$, $t \in [0, 5]$

MMsRDTM		AHRDTM		
Processing time (s)	Time-step, R	Admissible local error, δ	Processing time (s)	Time-step, R
		0.1	0.875	3
1443.700	5000	0.01	1.156	4
		0.001	1.688	6

Table 4
The duration of processing time and the total numbers of time-step for $\tau = 1.0$, $h = 0.0001$, $t \in [0, 5]$

MMsRDTM		AHRDTM		
Processing time (s)	Time-step, R	Admissible local error, δ	Processing time (s)	Time-step, R
		0.1	0.780	3
14000.000	50000	0.01	1.076	4
		0.001	1.655	6

The comparison through graphical illustrations of the approximate solution AHRDTM with $\delta = 0.1$, AHRDTM with $\delta = 0.01$, and the exact solution for $t \in [8.5, 9.5]$ and $x \in [-5, 5]$, which involves the real part and imaginary part, are shown in Figure 1a, and Figure 1b respectively. The comparisons in Figure 1a and Figure 1b clearly show that the graphs of the AHRDTM with $\delta = 0.01$ have similar shape and size with the exact solutions than the graph of AHRDTM with $\delta = 0.1$. Thus, smaller value of δ gives better accuracy in approximating solutions. The AHRDTM solutions for this sort of NLSE are therefore proved to be approximately accurate near to the exact solutions.



(a) (b)

Fig. 1. The graphs shown in Figure 1a and Figure 1b are the exact solutions, and the AHRDTM with $\delta = 0.1$, and AHRDTM with $\delta = 0.01$ involving the real and imaginary part respectively

Example 2. We considered NLSE with trapping potential of the form [25]

$$iu_t + \frac{1}{2}u_{xx} - ucos^2(x) - |u|^2 u = 0, \quad (23)$$

with initial condition

$$u(x,0) = \sin(x) \quad (24)$$

$\sin(x)e^{(-\frac{3i}{2}t)}$ is this equation's exact solution. By applying the AHRDTM to equation (23) and using fundamental properties of AHRDTM, we have

$$U_{k+1,\mathrm{r}}(x) = \left(\frac{I}{k+1}\right)\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}U_k(x) - U_k(x)cos^2(x) - A_{k,r}\right). \quad (25)$$

We write the transformed initial condition (24) as

$$U_0(x) = \sin(x) \quad (26)$$

The adaptive algorithm is then applied to get accurate approximate solution for this example.

The performance analysis is summarized in Table 5, Table 6, Table 7, and Table 8. The tables present comparisons of the processing time and the time-step, R used in solving equation (23) by the MMsRDTM, and the AHRDTM at N = 6, with respect to δ over the interval $[0, 3]$.

Table 5, Table 6, Table 7, and Table 8 show that AHRDTM is more efficient than the classical MMsRDTM. This is based on observations of the processing time and number of time-steps needed in approximating solutions for this problem. By observing the processing time based on the different values of h for each table presented in the tables, the demonstrated adaptive approach is proven to be highly rapid and efficient.

Table 5
The duration of processing time and the total numbers of time-step for $\tau = 1.0$, $h = 0.1$, $t \in [0, 3]$

MMsRDTM		AHRDTM		
Processing time (s)	Time-step, R	Admissible local error, δ	Processing time (s)	Time-step, R
		0.1	5.577	3
104.145	30	0.01	7.921	4
		0.001	10.546	5

Table 6
The duration of processing time and the total numbers of time-step for $\tau = 1.0$, $h = 0.01$, $t \in [0, 3]$

MMsRDTM		AHRDTM		
Processing time (s)	Time-step, R	Admissible local error, δ	Processing time (s)	Time-step, R
		0.1	5.483	3
1077.400	300	0.01	8.030	4
		0.001	10.751	5

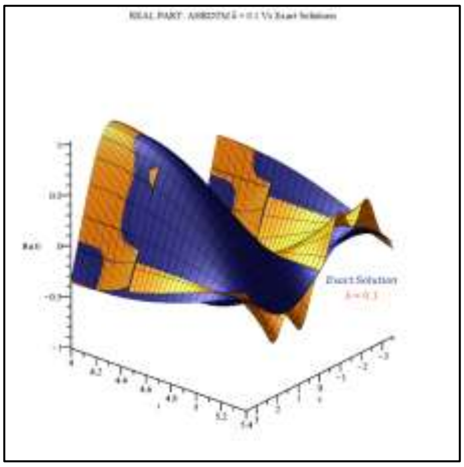
Table 7
The duration of processing time and the total numbers of time-step for $\tau = 1.0$, $h = 0.001$, $t \in [0, 3]$

MMsRDTM		AHRDTM		
Processing time (s)	Time-step, R	Admissible local error, δ	Processing time (s)	Time-step, R
		0.1	5.515	3
10890.500	3000	0.01	8.249	4
		0.001	10.937	5

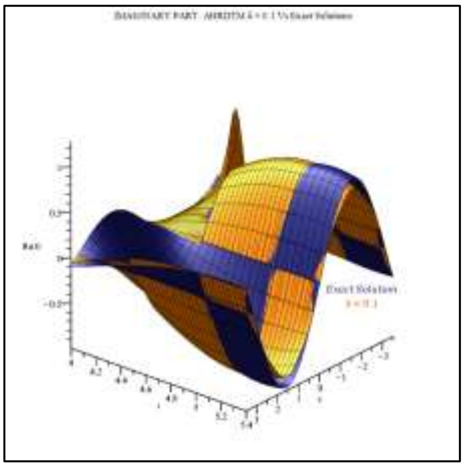
Table 8
The duration of processing time and the total numbers of time-step for $\tau = 1.0$, $h = 0.0001$, $t \in [0, 3]$

MMsRDTM		AHRDTM		
Processing time (s)	Time-step, R	Admissible local error, δ	Processing time (s)	Time-step, R
		0.1	5.640	3
106560.000	30000	0.01	8.219	4
		0.001	11.171	5

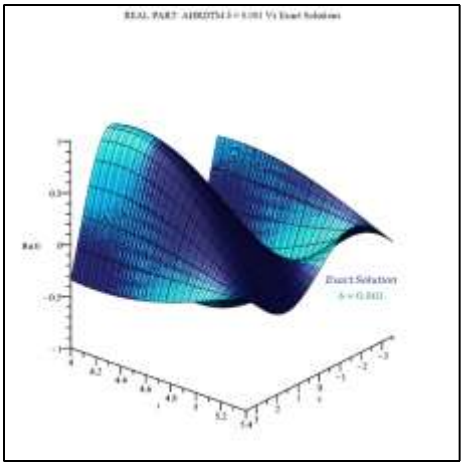
The comparison by graphical pictorials of approximate solutions AHRDTM with $\delta = 0.1$, AHRDTM with $\delta = 0.001$, and the exact solution for $t \in [4, 5.4]$ and $x \in [-3.5, 3.5]$, which involve the real and imaginary part, are shown in Figure 2a, Figure 2b, Figure 2c, Figure 2d, Figure 2e, and Figure 2f. Figure 2c and Figure 2d shows that the graphs of the AHRDTM with $\delta = 0.001$ are similar with their exact solutions than the graph of AHRDTM with $\delta = 0.1$ as shown in Figure 2a and Figure 2b. Thus, smaller value of δ gives greater accuracy in approximating solutions. The AHRDTM solutions for this type of NLSE are therefore proved to be significantly near to the exact solutions.



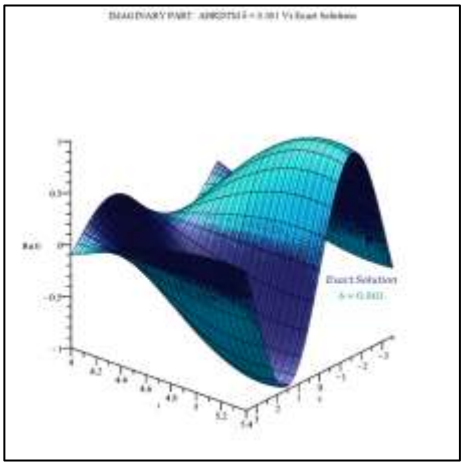
(a)



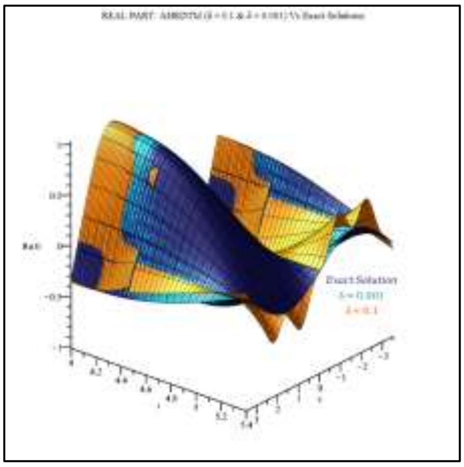
(b)



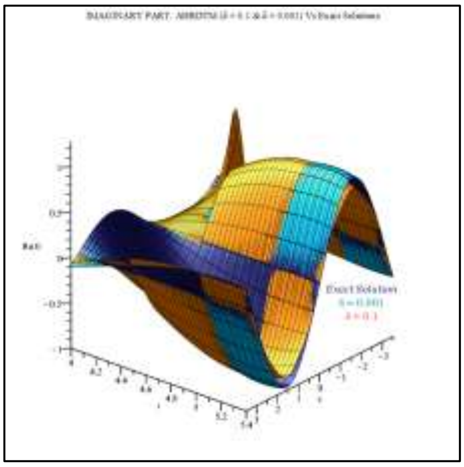
(c)



(d)



(e)



(f)

Fig. 2. The graphs shown in Figure 2a and Figure 2b are the exact solutions, and AHRDTM with $\delta = 0.1$, Figure 2c and Figure 2d are the exact solutions, and AHRDTM with $\delta = 0.001$, while Figure 2e and Figure 2f are the exact solutions, AHRDTM with $\delta = 0.1$, and AHRDTM with $\delta = 0.001$, which involve the real and imaginary part respectively

4. Conclusions

This paper introduces a novel, efficient, and accurate approach called the Adaptive Hybrid Reduced Differential Transform Method (AHRDTM), an adaptive approximation analytical method for handling NLSEs. The results illustrate the method's accuracy, effectiveness, and reliability, as evidenced by the numerical outcomes and graphical representations. Consequently, AHRDTM emerges as a valuable mathematical tool for solving NLSEs, providing solutions with high accuracy and demonstrating a notable superiority over MRDTM in terms of accuracy. All calculations in this paper were conducted using Maple 2021.

Acknowledgement

This research is supported financially by the University of Malaysia Sabah through Research Grant UMSGreat GUG0576-1/2023, contributing to the successful advancement of the project.

References

[1] Zhao, Tie-Hong, Miao-Kun Wang, and Yuming Chu. "Concavity and Bounds Involving Generalized Elliptic Integral of the First Kind." *Journal of Mathematical Inequalities* 15, no. 2 (2021): 701–724. https://doi.org/10.7153/jmi-2021-15-50

[2] Ghanbari, Behzad. "On the Modeling of an Eco-Epidemiological Model Using a New Fractional Operator." *Results in Physics* 21 (2021): 103799. https://doi.org/10.1016/j.rinp.2020.103799

[3] Owolabi, Kolade M. "Robust and Adaptive Techniques for Numerical Simulation of Nonlinear Partial Differential Equations of Fractional Order." *Communications in Nonlinear Science and Numerical Simulation* 44 (2017): 304–317. https://doi.org/10.1016/j.cnsns.2016.08.021

[4] El-Kalla, I. L., E. M. Mohamed, and H. A. A. El-Saka. "An Accelerated Solution for Some Classes of Nonlinear Partial Differential Equations." *Journal of the Egyptian Mathematical Society* 29, no. 1 (2021): 7. https://doi.org/10.1186/s42787-021-00116-9

[5] Özkan, Yeşim Sağlam, Emrullah Yaşar, and Aly R. Seadawy. "A third-order nonlinear Schrödinger equation: the exact solutions, group-invariant solutions and conservation laws." *Journal of Taibah University for Science* 14, no. 1 (2020): 585–597. https://doi.org/10.1080/16583655.2020.1760513

[6] Lü, Xing, and Si-Jia Chen. "Interaction Solutions to Nonlinear Partial Differential Equations via Hirota Bilinear Forms: One-Lump-Multi-Stripe and One-Lump-Multi-Soliton Types." *Nonlinear Dynamics* 103, no. 1 (2021): 947–977. https://doi.org/10.1007/s11071-020-06068-6

[7] Wazwaz, Abdul-Majid. "Padé Approximants and Adomian Decomposition Method for Solving the Flierl–Petviashivili Equation and Its Variants." *Applied Mathematics and Computation* 182, no. 2 (2006): 1812–1818. https://doi.org/10.1016/j.amc.2006.06.018

[8] Aldhabani, Musaad S., Kamsing Nonlaopon, S. Rezaei, Fatimah S.Bayones, S.K. Elagan, and Sobhy A.A. El-Marouf. "Abundant Solitary Wave Solutions to a Perturbed Schrödinger Equation with Kerr Law Nonlinearity via a Novel Approach." *Results in Physics* 35 (2022): 105385. https://doi.org/10.1016/j.rinp.2022.105385

[9] Nonlaopon, Kamsing, Shahram Rezapour, Musaad S. Aldhabani, Samirah H. Alsulami, and S.K. Elagan. "On an Efficient Approach to Solutions of a Perturbed Nonlinear Schrödinger's Equation." *Results in Physics* 39 (2022): 105738. https://doi.org/10.1016/j.rinp.2022.105738

[10] Gu, Yongyi, and Najva Aminakbari. "Bernoulli (G′/G)-Expansion Method for Nonlinear Schrödinger Equation with Third-Order Dispersion." *Modern Physics Letters B* 36, no. 11 (2022): 2250028. https://doi.org/10.1142/S0217984922500282

[11] Islam, Zahidul, Alrazi Abdeljabbar, Md. Abu Naim Sheikh, Harun-Or-Roshid, and M.A. Taher. "Optical Solitons to the Fractional Order Nonlinear Complex Model for Wave Packet Envelope." *Results in Physics* 43 (2022): 106095. https://doi.org/10.1016/j.rinp.2022.106095

[12] Kapoor, Mamta, Nehad Ali Shah, and Wajaree Weera. "Analytical Solution of Time-Fractional SchröDinger Equations via Shehu Adomian Decomposition Method." *AIMS Mathematics* 7, no. 10 (2022): 19562–19596. https://doi.org/10.3934/math.20221074

[13] Jawad, Huda, and Ali Al-Fayadh. "An Efficient Variational Homotopy Transform Method for Solving Schrodinger Equation." *Journal of Physics: Conference Series* 2322 (2022): 012044. https://doi.org/10.1088/1742-6596/2322/1/012044

[14] Keskin, Yildiray, and Galip Oturanç. "Reduced Differential Transform Method for Partial Differential Equations." *International Journal of Nonlinear Sciences and Numerical Simulation* 10, no. 6 (2009): 741–749. https://doi.org/10.1515/IJNSNS.2009.10.6.741

[15] Deresse, Alemayehu Tamirie. "Analytical Solution of One-Dimensional Nonlinear Conformable Fractional Telegraph Equation by Reduced Differential Transform Method." *Advances in Mathematical Physics* 2022 (2022): 7192231. https://doi.org/10.1155/2022/7192231

[16] Ray, S Saha. "Numerical Solutions and Solitary Wave Solutions of Fractional KDV Equations Using Modified Fractional Reduced Differential Transform Method." *Computational Mathematics and Mathematical Physics* 53, no. 12 (2013): 1870–1881. https://doi.org/10.1134/S0965542513120142

[17] Odibat, Zaid M., Cyrille Bertelle, M. A. Aziz-Alaoui, and Gérard H.E. Duchamp. "A Multi-Step Differential Transform Method and Application to Non-Chaotic or Chaotic Systems." *Computers and Mathematics with Applications* 59, no. 4 (2010): 1462–1472. https://doi.org/10.1016/j.camwa.2009.11.005

[18] Al-Smadi, Mohammed, Asad Freihat, Hammad Khalil, Shaher Momani, and Rahmat Ali Khan. "Numerical Multistep Approach for Solving Fractional Partial Differential Equations." *International Journal of Computational Methods* 14, no. 03 (2017): 1750029. https://doi.org/10.1142/S0219876217500293

[19] Hussin, Che Haziqah Che, Adem Kilicman, and Amirah Azmi. "Analytical Solutions of Nonlinear Schrodinger Equations Using Multistep Modified Reduced Differential Transform Method." *COMPUSOFT, An International Journal of Advanced Computer Technology* 7, no. 11 (2018): 2939–2944.

[20] Sabdin, Abdul Rahman Farhan, Che Haziqah Che Hussin, Arif Mandangan, Graygorry Brayone Ekal, and Jumat Sulaiman. "Approximate Analytical Solutions for Non-Linear Telegraph Equations with Source Term." *Journal of Advanced Research in Applied Sciences and Engineering Technology* 31, no. 3 (2023): 238–248. https://doi.org/10.37934/araset.31.3.238248

[21] Sabdin, Abdul Rahman Farhan, Che Haziqah Che Hussin, Graygorry Brayone Ekal, Arif Mandangan, and Jumat Sulaiman. "Approximate Analytical Solution for Time-Fractional Nonlinear Telegraph Equations with Source Term." *Journal of Advanced Research in Applied Sciences and Engineering Technology* 31, no. 1 (2023): 132–143. https://doi.org/10.37934/araset.31.1.132143

[22] Gökdoğan, Ahmet, Mehmet Merdan, and Ahmet Yildirim. "Adaptive Multi-Step Differential Transformation Method to Solving Nonlinear Differential Equations." *Mathematical and Computer Modelling* 55, no. 3–4 (2012): 761–769. https://doi.org/10.1016/j.mcm.2011.09.001

[23] El-Zahar, Essam R. "Applications of Adaptive Multi Step Differential Transform Method to Singular Perturbation Problems Arising in Science and Engineering." *Applied Mathematics & Information Sciences* 9, no. 1 (2015): 223–232. https://doi.org/10.12785/amis/090128

[24] Taheri, Alireza Ghomi, Farbod Setoudeh, Mohammad Bagher Tavakoli, and Esmaeil Feizi. "Nonlinear Analysis of Memcapacitor-Based Hyperchaotic Oscillator by Using Adaptive Multi-Step Differential Transform Method." *Chaos, Solitons & Fractals* 159 (2022): 112122. https://doi.org/10.1016/j.chaos.2022.112122

[25] Ravi Kanth, A.S.V., and K. Aruna. "Two-Dimensional Differential Transform Method for Solving Linear and Non-Linear Schrödinger Equations." *Chaos, Solitons & Fractals* 41, no. 5 (2009): 2277–2281. https://doi.org/10.1016/j.chaos.2008.08.037