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A Homotopy Analysis Method for a Coupled Two-Dimensional Inverse Time-Fractional Navier–Stokes Problem with a Moving Boundary

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ABSTRACT

This study develops a semi-analytical framework for a coupled two-dimensional inverse time-fractional incompressible Navier-Stokes moving-boundary problem. The governing system consists of two coupled momentum equations involving a Caputo time-fractional derivative, nonlinear convective terms, a pressure field, viscous diffusion, and a time-dependent unknown coefficient multiplying prescribed spatial-temporal forcing functions. The problem is posed on a moving physical domain whose upper boundary is represented by an unknown interface $s(x,t)$. In addition to the initial and boundary conditions, an overdetermined condition is imposed to recover the unknown coefficient $g(t)$. The Homotopy Analysis Method (HAM) is used to construct approximate series solutions for the velocity components $u(x,y,t)$ and $v(x,y,t)$, while the pressure $p(x,y,t)$ is recovered from the momentum equations, and the moving boundary is obtained through the vertical velocity condition on the interface. Exact solutions are used to validate the proposed method and to provide explicit benchmark expressions for the velocity field, pressure, moving boundary, source terms, and unknown coefficient. Numerical comparisons show that the second-order HAM approximation provides stable and accurate recovery of the horizontal velocity, pressure field, moving boundary, and unknown coefficient. Although the vertical velocity component exhibits larger relative errors due to the small magnitude of the exact solution, the approximation remains bounded and stable over the time interval. The results demonstrate that the proposed HAM-based approach provides an effective semi-analytical tool for solving coupled inverse fractional fluid-flow problems on moving domains.

1. Introduction

The Navier-Stokes equations remain one of the central mathematical models for describing the motion of viscous incompressible fluids. They model the conservation of mass and momentum and are applied in real-world scenarios, including free-surface flows, biological fluids, porous-media transport, melt dynamics, and fluid-structure interaction. Because of their nonlinear convective structure, pressure coupling, incompressibility constraint, and sensitivity to boundary conditions, the

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Navier-Stokes equations continue to motivate substantial analytical, numerical, and semi-analytical research.

Classical free-boundary and moving-boundary problems have been studied extensively in heat transfer phase change, and viscous fluid mechanics, as seen in Crank [1], Rubinstein [2], Meirmanov [3], and Bae [4]. Unlike classical integer-order derivatives, fractional derivatives incorporate temporal effects, making them useful for systems in which the current state depends not only on the present configuration but also on the previous history of the process. This makes fractional models useful in systems involving anomalous diffusion, porous media, viscoelasticity, rheology, and other processes with hereditary effects. Zhou and Peng [5] studied Navier-Stokes equations involving a Caputo time-fractional derivative and emphasized their relevance for modeling anomalous diffusion in fractal media. Their work established local and global mild solutions and regularity results for time-fractional Navier-Stokes equations, thereby providing an important theoretical foundation for Caputo-type fractional fluid models.

The analytical theory of time-fractional Navier-Stokes equations has also been extended to systems with delayed external forces. Alam and Dubey [6], considered time-fractional Navier-Stokes equations driven by finite delayed forces on a bounded domain and transformed the governing system into an abstract Cauchy problem. They established local existence and uniqueness of mild solutions, together with global continuation and regularity results, using semigroup theory, fractional calculus, and the Banach contraction principle. Their work is relevant because it shows how time-fractional Navier-Stokes models can incorporate hereditary forcing mechanisms, where the external force depends not only on the present state but also on the history of the velocity field.

From the computational side, Abedini et al. [7] developed a method of lines approach for two-dimensional unsteady incompressible Navier-Stokes equations with a Caputo time-fractional derivative. Their formulation used the stream function-vorticity form of the equations, finite-difference spatial discretization, and fractional backward differentiation formulas for the resulting fractional differential system. A significant aspect of their study is that the velocity field was treated as a function of two spatial variables, allowing the flow behavior to be examined through streamlines and vorticity contours in a lid-driven cavity problem. Their work highlights both the importance and difficulty of two-dimensional time-fractional incompressible flow, particularly when fractional memory, pressure coupling, and incompressibility are present.

Several semi-analytical and transform-based methods have also been proposed for time-fractional Navier-Stokes equations. Sharma et al. [8] applied a Variational Iteration Transform Method to the n-dimensional time-fractional Navier-Stokes equation with the fractional derivative described in the Caputo sense. Their work emphasizes that time-fractional Navier-Stokes equations generalize the classical Navier-Stokes model by replacing the first-order time derivative, and that such models are relevant in fluid dynamics, geophysics, and engineering. They also state that semi-analytical methods are attractive because exact solutions are difficult to obtain, and fully numerical procedures may require many computational steps.

Sawant and Deshpande [9] proposed another semi-analytical approach for time-fractional Navier-Stokes equations by combining the Adomian Decomposition Method with the Kamal integral transform. Their method was applied to time-fractional Navier-Stokes equations to obtain analytical series-type solutions, and illustrative examples were used to validate the method against existing solution techniques. This work reflects the broader trend of using decomposition, transform, and homotopy-based methods to obtain explicit or semi-explicit approximations for fractional fluid-flow models.

Inverse Navier-Stokes problems form another important motivation for the present study. In many applications, the governing equations may be known, but certain quantities, such as an initial state, forcing coefficient, source strength, viscosity, or boundary behavior, must be recovered from additional observations. O'Connor et al. [10] studied iterative methods for retrospective Navier-Stokes inverse problems and emphasized that problems can be ill-conditioned even when the corresponding forward model is solvable. They compared direct adjoint looping, simple backward, and the quasi-reversible method (QRM) for inverse problems constrained by the Korteweg-de Vries-Burgers (kdVB) equation and by two-dimensional incompressible Navier-Stokes flow. Their results showed that simple backward integration and the quasi-reversible method (QRM) can outperform direct adjoint looping in recovering target states for the tested Navier-Stokes inverse problem. This illustrates the difficulty of inverse recovery in nonlinear fluid systems.

Despite these developments, the existing literature largely focuses on one of several separate directions: theoretical existence and regularity of time-fractional Navier-Stokes equations, fixed-domain numerical simulation of fractional incompressible flow, semi-analytical solutions for standard fractional Navier-Stokes models, or inverse Navier-Stokes problems on fixed domains. Much less attention has been given to coupled inverse time-fractional Navier-Stokes problems on moving domains, particularly problems in which the velocity field, pressure, moving boundary, and an unknown time-dependent coefficient must be recovered simultaneously.

The present work is motivated by the need for semi-analytical methods capable of treating inverse, coupled, nonlinear, time-fractional fluid-flow problems on moving domains. In the model presented in this study, the unknowns are the horizontal velocity $u(x, y, t)$, vertical velocity $v(x, y, t)$, pressure $p(x, y, t)$, moving boundary $s(x, t)$, and time-dependent coefficient $g(t)$. The governing systems consists of two coupled time-fractional Navier-Stokes momentum equations with Caputo time derivatives, nonlinear convection terms, viscous diffusion, pressure gradients, and prescribed forcing profiles multiplied by the unknown coefficient $g(t)$. The incompressibility condition is imposed, and the moving boundary is recovered through the vertical condition on the interface. An additional condition is used to identify the unknown coefficient $g(t)$.

To solve the problem, this study employs the Homotopy Analysis Method (HAM), originally developed by Liao [11,12]. HAM is attractive because it does not require small physical parameters and introduces convergence-control parameters that can be adjusted to influence the convergence of the approximation series. These parameters are computed from prescribed right-boundary velocity data rather than from classical h-curves. This boundary-driven choice is consistent with the physical data of the problem and improves the reliability of the approximation.

The main contributions of this paper are as follows. First, a coupled inverse time-fractional Navier–Stokes moving-boundary problem is formulated in which the velocity field, pressure, moving interface, and unknown time-dependent coefficient are recovered simultaneously. Second, a HAM-based semi-analytical scheme is developed directly on the moving physical domain. Third, time-dependent convergence-control parameters are determined from boundary data rather than from conventional *h-curves*. Finally, relative error tables are presented for $\alpha = 0.75$, together with a pressure-sensitivity comparison for $\alpha = 0.95$.

2. Governing Equations

Let the time-dependent domain be

$$\Omega(t) = \{(x, y): 0 < x < 1, 0 < y < s(x, t), 0 < t \leq T\}, \tag{1}$$

where $s(x, t)$ denotes the moving upper boundary. The unknown quantities are

$$u(x, y, t), v(x, y, t), p(x, y, t), s(x, t), g(t), \tag{2}$$

The coupled two-dimensional inverse time-fractional incompressible Navier-Stokes system is

$$\rho^C D_t^\alpha u + \rho(uu_x + vu_y) = -p_x + \mu(u_{xx} + u_{yy}) + \rho g(t) F_1(x, y, t), \tag{3}$$
$$\rho^C D_t^\alpha v + \rho(uv_x + vv_y) = -p_y + \mu(v_{xx} + v_{yy}) + \rho g(t) F_2(x, y, t), \tag{4}$$
with the incompressibility condition
$$u_x + v_y = 0. \tag{5}$$
Here $0 < \alpha < 1, \rho > 0$ is the density, $\mu > 0$ is the viscosity, the Caputo derivative is defined by
$$^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau. \tag{6}$$
For the proposed benchmark problem, the forcing functions are
$$F_1(x, y, t) = (1+t) y (1 + 2x + y), \tag{7}$$
$$F_2(x, y, t) = (1+t) x (1 + x + 2y), \tag{8}$$
The exact velocity components are
$$u(x, y, t) = (1 + \delta t)(x + y), \tag{9}$$
$$v(x, y, t) = (1 + \delta t)(x - y), \tag{10}$$
$$g(t) = 1 + \delta t, \tag{11}$$

The exact solution of the moving boundary is
$$s(x, t) = e^{-(1+\lambda)\left(t + \frac{\delta t^2}{2}\right)} + \lambda x, \ \lambda = \sqrt{2} - 1, \tag{12}$$
The compatible pressure field is
$$p(x, y, t) = \rho\left[(1 + \delta t)(1 + t) xy (1 + x + y) - (1 + \delta t)^2 (x^2 + y^2) + \frac{\delta t^{1-\alpha}}{\Gamma(2-\alpha)}\left(\frac{y^2}{2} - \frac{x^2}{2} - xy\right)\right]. \tag{13}$$
Although $u(x, y, t), v(x, y, t), s(x, t)$, and $g(t)$ do not explicitly contain α, the model remains dependent on α through the Caputo derivative. This dependence appears in the compatible pressure field through $\delta t^{1-\alpha}/\Gamma(2-\alpha)$ and also affects the HAM corrections terms, the convergence-control parameters, and pressure recovery.

Initial Conditions At $t = 0$, the initial velocity components are
$$u(x, y, 0) = x + y, \quad v(x, y, 0) = x - y, \tag{14}$$
for $0 < x < 1$ and $0 < y < s(x, 0)$. The initial pressure is
$$p(x, y, 0) = \rho[xy(1 + x + y) - x^2 - y^2], \tag{15}$$
The initial coefficient value is $g(0) = 1$. (16)
and the initial moving boundary is
$$s(x, 0) = 1 + \lambda x. \tag{17}$$
Boundary Conditions

On $x = 0$,
$$u(0, y, t) = (1 + \delta t) y, \quad v(0, y, t) = -(1 + \delta t) y. \tag{18}$$

Since pressure is recovered from a pressure-gradient relation, a pressure reference is required. The left-boundary pressure is given as:

$$p(0, y, t) = \rho\left[-(1 + \delta t)^2 y^2 + \frac{\delta t^{1-\alpha} y^2}{2\Gamma(2-\alpha)}\right]. \tag{19}$$

On $x = 1$,

$$u(1, y, t) = (1 + \delta t)(1 + y), \quad v(1, y, t) = (1 + \delta t)(1 - y). \tag{20}$$

On $y = 0$,

$$u(x,0,t) = (1+\delta t)x, \quad v(x,0,t) = (1+\delta t)x. \tag{21}$$

On the moving upper boundary $y = s(x,t)$,

$$u(x,s(x,t),t) = (1+\delta t)[x + s(x,t)], \tag{22}$$

$$v(x,s(x,t),t) = (1+\delta t)[x - s(x,t)], \tag{23}$$

Overdetermined Condition

$$u\left(\frac{1}{2},\frac{1}{2},t\right) = 1 + \delta t, \tag{24}$$

3. Application of the Homotopy Analysis Method (HAM)

Consider the coupled two-dimensional inverse time-fractional Navier-Stokes moving boundary problems written in residual form as

$$\rho\, {}^{C}D_t^{\alpha} u + \rho\big(uu_x + vu_y\big) + p_x - \mu\big(u_{xx} + u_{yy}\big) - \rho g(t) F_1(x,y,t) = 0, \tag{25}$$

$$\rho\, {}^{C}D_t^{\alpha} v + \rho\big(uv_x + vv_y\big) + p_y - \mu\big(v_{xx} + v_{yy}\big) - \rho g(t) F_2(x,y,t) = 0, \tag{26}$$

Following the basic idea of HAM, let $\phi_1(x,y,t;q)$and $\phi_2(x,y,t;q)$ be the homotopy functions corresponding to $u(x,y,t)$ and $v(x,y,t)$, respectively, where $q \in [0,1]$ is the embedding parameter. When $q = 0$, the homotopy functions reduce to the initial guesses, and when $q = 1$, the desired solutions are recovered:

$$\phi_1(x,y,t;0) = u_0(x,y,t), \;\; \phi_2(x,y,t;0) = v_0(x,y,t), \tag{27}$$

$$\phi_1(x,y,t;1) = u(x,y,t), \;\; \phi_2(x,y,t;1) = v(x,y,t), \tag{28}$$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be auxiliary linear operators satisfying

$$\mathcal{L}_1[C_1] = 0, \;\; \mathcal{L}_2[C_2] = 0, \tag{29}$$

where C_1 and C_2 are constants determined from the imposed initial conditions. In the current time-fractional formulation, the inverse operators are chosen in terms of the Riemann-Liouville fractional integral

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau. \tag{30}$$

Define the nonlinear operators

$$\aleph_1[\phi_1,\phi_2,p,g] = \rho\, {}^{C}D_t^{\alpha}\phi_1 + \rho\left(\phi_1\frac{\partial\phi_1}{\partial x} + \phi_2\frac{\partial\phi_1}{\partial y}\right) + \frac{\partial p}{\partial x} - \mu\left(\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2}\right) - \rho g(t) F_1(x,y,t) \tag{31}$$

$$\aleph_2[\phi_1,\phi_2,p,g] = \rho\, {}^{C}D_t^{\alpha}\phi_2 + \rho\left(\phi_1\frac{\partial\phi_2}{\partial x} + \phi_2\frac{\partial\phi_2}{\partial y}\right) + \frac{\partial p}{\partial y} - \mu\left(\frac{\partial^2\phi_2}{\partial x^2} + \frac{\partial^2\phi_2}{\partial y^2}\right) - \rho g(t) F_2(x,y,t) \tag{32}$$

The zeroth-order deformation equations are constructed as

$$(1-q)\mathcal{L}_1[\phi_1(x,y,t;q) - u_0(x,y,t)] = q h_1 H_1(x,y,t)\aleph_1[\phi_1,\phi_2,p,g], \tag{33}$$

$$(1-q)\mathcal{L}_2[\phi_2(x,y,t;q) - v_0(x,y,t)] = q h_2 H_2(x,y,t)\aleph_2[\phi_1,\phi_2,p,g], \tag{34}$$

where h_1 and h_2 are convergence control parameters, and $H_1(x,y,t)$ and $H_2(x,y,t)$ are nonzero auxiliary functions. In this study, the auxiliary functions were chosen as

$H_1(x,y,t) = 1, \qquad H_2(x,y,t) = 1.$ (35)

When $q = 0$, the zeroth-order deformation equations give the initial approximations. When $q = 1$, the deformation equations become equivalent to the original coupled time-fractional Navier-Stokes equations.

Assuming that ϕ_1 and ϕ_2 are analytic in q, they are expanded in Maclaurin series as

$\phi_1(x,y,t;q) = u_0(x,y,t) + \sum_{m=1}^{\infty} u_m(x,y,t)q^m,$ (36)

$\phi_2(x,y,t;q) = v_0(x,y,t) + \sum_{m=1}^{\infty} v_m(x,y,t)q^m,$ (37)

where

$u_m(x,y,t) = \frac{1}{m!}\frac{\partial^m \phi_1(x,y,t;q)}{\partial q^m}\Big|_{q=0}$, and $v_m(x,y,t) = \frac{1}{m!}\frac{\partial^m \phi_2(x,y,t;q)}{\partial q^m}\Big|_{q=0}$ (38)

If the series converges at $q = 1$, then the approximate solutions are

$u_{HAM}(x,y,t) = \sum_{m=0}^{N} u_m(x,y,t), \qquad v_{HAM}(x,y,t) = \sum_{m=0}^{N} v_m(x,y,t)$. (39)

Similarly,

$p_{HAM}(x,y,t) = \sum_{m=0}^{N} p_m(x,y,t), \quad g_{HAM}(t) = \sum_{m=0}^{N} g_m(t).$ (40)

For the numerical computations in this study, the HAM series is truncated at $N = 2$. (41)

This choice was made because the second-order approximation produced the most stable full-system approximation for the coupled velocity, pressure, moving boundary, and coefficient recovery. Higher-order approximations were found to increase computational instability in the recovered pressure terms.

For the present model, the initial guesses are selected from the initial data:

$u_0(x,y,t) = x + y,$ (42)

$v_0(x,y,t) = x - y,$ (43)

and

$p_0(x,y,t) = \rho[xy(1 + x + y) - x^2 - y^2].$ (44)

The switching parameter is defined by

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{45}$$

The nonlinear terms are expanded using Cauchy products. For the *u-momentum* equation,

$$\varsigma_m^{(u)}(x,y,t) = \sum_{k=0}^{m-1}\left[u_k(x,y,t)\frac{\partial u_{m-1-k}(x,y,t)}{\partial x} + v_k(x,y,t)\frac{\partial u_{m-1-k}(x,y,t)}{\partial y}\right]. \tag{46}$$

For the *v-momentum* equation,

$$\varsigma_m^{(v)}(x,y,t) = \sum_{k=0}^{m-1}\left[u_k(x,y,t)\frac{\partial v_{m-1-k}(x,y,t)}{\partial x} + v_k(x,y,t)\frac{\partial v_{m-1-k}(x,y,t)}{\partial y}\right]. \tag{47}$$

The *mth-order* residuals are

$$R_m^{(u)}(x,y,t) = \rho^C D_t^\alpha u_{m-1}(x,y,t) + \rho\varsigma_m^{(u)}(x,y,t) + \frac{\partial p_{m-1}(x,y,t)}{\partial x} - \mu\left(\frac{\partial^2 u_{m-1}(x,y,t)}{\partial x^2} + \frac{\partial^2 u_{m-1}(x,y,t)}{\partial y^2}\right) - \rho g_{m-1}(t)F_1(x,y,t) \tag{48}$$

and

$$R_m^{(v)}(x,y,t) = \rho^C D_t^\alpha v_{m-1}(x,y,t) + \rho\varsigma_m^{(v)}(x,y,t) + \frac{\partial p_{m-1}(x,y,t)}{\partial y} - \mu\left(\frac{\partial^2 v_{m-1}(x,y,t)}{\partial x^2} + \frac{\partial^2 v_{m-1}(x,y,t)}{\partial y^2}\right) - \rho g_{m-1}(t)F_2(x,y,t) \tag{49}$$

Applying the inverse fractional operator gives the *m-th order* deformation equations

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + h_1 I^\alpha\left[R_m^{(u)}(x,y,t)\right], \tag{50}$$

and

$$v_m(x,y,t) = \chi_m v_{m-1}(x,y,t) + h_2 I^\alpha\left[R_m^{(v)}(x,y,t)\right], \tag{51}$$

Therefore, the HAM approximations are obtained recursively by computing $u_0, u_1, \dots, u_N$ and $v_0, v_1, \dots, v_N$.

4. HAM Solution to the Proposed Problem

When using HAM, the initial approximations are chosen as

$$u_0(x,y,t) = x + y, \quad v_0(x,y,t) = x - y, \tag{52}$$

and

$$p_0(x,y,t) = \rho[xy(1+x+y) - x^2 - y^2]. \tag{53}$$

The unknown coefficient is recovered for the overdetermined condition by setting

$$u_{HAM}\left(\frac{1}{2},\frac{1}{2},t\right) = u\left(\frac{1}{2},\frac{1}{2},t\right), \tag{54}$$

thereby giving us

$$g_{HAM}(t) = 1 + \delta t. \tag{55}$$

The moving boundary is recovered from

$$v_{HAM}(x, s_{HAM}(x,t), t) = v(x, s(x,t), t), \tag{56}$$

The pressure is recovered from the *x-momentum* equation using

$$(p_{HAM}(x,y,t))_x = -\rho^C D_t^\alpha u_{HAM}(x,y,t) - \rho\left[u_{HAM}(x,y,t)\big(u_{HAM}(x,y,t)\big)_x + v_{HAM}(x,y,t)\big(u_{HAM}(x,y,t)\big)_y\right] + \mu\left[\big(u_{HAM}(x,y,t)\big)_{xx} + \big(u_{HAM}(x,y,t)\big)_{yy}\right] + \rho g_{HAM}(t)F_1(x,y,t),$$

(57)

with

$$p_{HAM}(x,y,t) = p(0,y,t) + \int_0^x \left(p_{HAM}(x,y,t)\right)_x\Big|_{x=\xi} d\xi. \tag{58}$$

5. Analysis Of Results

Using HAM approximations, we obtain the following error analysis for $\rho = 1, \mu = 1, x = 0.85, y = 0.75,\ \delta = 0.20$. (59)

The Relative Error is computed by

$$RelErr_Q = \left|\frac{Q_{HAM} - Q_{exact}}{Q_{exact}}\right|, \tag{60}$$

where Q represents u, v, p, s, and g.

Unlike the conventional HAM approach, where h_1 and h_2 are selected using *h-curves* or residual minimization, this present study determines them at each time step from prescribed right-boundary velocity data. Since

$$u(1,y,t) = (1+\delta t)(1+y), \quad v(1,y,t) = (1+\delta t)(1-y), \tag{61}$$

The boundary -collocation level $y = 0.75$ gives

$$u_{HAM}(1,0.75,t;h_1,h_2) = 1.75(1+\delta t), \tag{62}$$

$$v_{HAM}(1,0.75,t;h_1,h_2) = 0.25(1+\delta t). \tag{63}$$

These equations are solved at each time level to obtain $h_1(t)$ and $h_2(t)$. Therefore, the convergence-control parameters are determined from prescribed boundary data rather than arbitrary *h-curves*.

In this section, an error analysis is carried out to evaluate the accuracy, convergence behavior, and computational performance of HAM.

Table 1
Time-dependent convergence control parameters when $\alpha = 0.75$

t	$h_1(t)$	$h_2(t)$
0	-1	-1
0.1	-0.562767003	-0.064603355
0.2	-0.334623262	-0.038413385
0.3	-0.246881009	-0.028340932
0.4	-0.198968182	-0.022840735
0.5	-0.16830659	-0.01932091
0.6	-0.146796326	-0.016851619
0.7	-0.130769088	-0.015011758
0.8	-0.118307189	-0.013581182
0.9	-0.10830456	-0.012432921
1.0	-0.100075697	-0.011488282

Table 2
Error Analysis for $u_{HAM}(x,y,t)$ when $x = 0.85, y = 0.75, \alpha = 0.75$

t	$u_{HAM}(x,y,t)$	$u(x,y,t)$	$RelErr_u$
0	1.6	1.6	0
0.1	1.6322	1.632	0.000123
0.2	1.6644	1.664	0.00024
0.3	1.6966	1.696	0.000354
0.4	1.7288	1.728	0.000463
0.5	1.761	1.76	0.000568
0.6	1.7932	1.792	0.00067
0.7	1.8254	1.824	0.000768
0.8	1.8576	1.856	0.000862
0.9	1.8898	1.888	0.000953
1.0	1.922	1.92	0.001042

Table 3
Error Analysis for $v_{HAM}(x,y,t)$ when $x = 0.85, y = 0.75, \alpha = 0.75$

t	$v_{HAM}(x,y,t)$	$v(x,y,t)$	$RelErr_v$
0	0.1	0.1	0
0.1	0.104068	0.102	0.020273
0.2	0.108136	0.104	0.039766
0.3	0.112204	0.106	0.058524
0.4	0.116271	0.108	0.076587
0.5	0.120339	0.11	0.093994
0.6	0.124407	0.112	0.110778
0.7	0.128475	0.114	0.126974
0.8	0.132543	0.116	0.142611
0.9	0.136611	0.118	0.157718
1.0	0.140679	0.12	0.172321

Table 4
Error Analysis for $p_{HAM}(x,y,t)$ when $x = 0.85, y = 0.75, \alpha = 0.75$

t	$p_{HAM}(x,y,t)$	$p(x,y,t)$	$RelErr_p$
0	0.372508	0.372481	0.0000722
0.1	0.390419	0.433772	0.099943
0.2	0.536569	0.57283	0.063302
0.3	0.697439	0.72304	0.035409
0.4	0.867931	0.88141	0.015293
0.5	1.046268	1.046896	0.0006
0.6	1.231599	1.218998	0.010337
0.7	1.423447	1.397437	0.018613
0.8	1.621512	1.582036	0.024953
0.9	1.825595	1.772678	0.029851
1.0	2.035555	1.969282	0.033653

Table 5
Error Analysis for $s_{HAM}(x,t)$ when $x = 0.85, \alpha = 0.75$

t	$s_{HAM}(x,t)$	$s(x,t)$	$RelErr_s$
0	1.352082	1.352082	0
0.1	1.231595	1.218978	0.01035
0.2	1.121467	1.101469	0.018156
0.3	1.02112	0.998058	0.023107
0.4	0.929952	0.907345	0.024916
0.5	0.847346	0.828022	0.023337
0.6	0.772682	0.758879	0.018189
0.7	0.705346	0.698799	0.00937
0.8	0.644736	0.646757	0.003125
0.9	0.590266	0.601819	0.019197
1.0	0.541376	0.563137	0.038642

Table 6
Error Analysis for $g_{HAM}(t)$ when $\alpha = 0.75$

t	$g_{HAM}(t)$	$g(t)$	$RelErr_g$
0	1	1	0
0.1	1.02	1.02	0
0.2	1.04	1.04	0
0.3	1.06	1.06	0
0.4	1.08	1.08	0
0.5	1.1	1.1	0
0.6	1.12	1.12	0
0.7	1.14	1.14	0
0.8	1.16	1.16	0
0.9	1.18	1.18	0
1.0	1.2	1.2	0

Table 7
Error Analysis for $p_{HAM}(x,y,t)$ when $x = 0.85, y = 0.75$ and $\alpha = 0.95$

t	$h_1(t)$	$h_2(t)$	$p_{HAM}(x,y,t)$	$p(x,y,t)$	$RelErr_p$
0	-1	-1	0.38203	0.348187	0.097198
0.1	-1.05963	-0.12164	0.242309	0.391426	0.380957
0.2	-0.5485	-0.06297	0.406582	0.542696	0.250809
0.3	-0.37315	-0.04284	0.580337	0.701415	0.17262
0.4	-0.28392	-0.03259	0.761125	0.866511	0.121622
0.5	-0.22968	-0.02637	0.948162	1.037641	0.086233
0.6	-0.19316	-0.02217	1.141094	1.214648	0.060555
0.7	-0.16684	-0.01915	1.339731	1.397449	0.041303
0.8	-0.14697	-0.01687	1.543956	1.585994	0.026505
0.9	-0.13141	-0.01509	1.753697	1.78025	0.014915
1.0	-0.11889	-0.01365	1.9689	1.980194	0.005703

From Table 1, for $\alpha = 0.75$, both $h_1(t)$ and $h_2(t)$ move toward zero as time increases, with $h_2(t)$ having a much smaller magnitude than $h_1(t)$. This suggests that the vertical velocity correction requires a more delicate convergence-control adjustment than the horizontal velocity correction.

The results shown in Table 2 show that the approximation for $u(x, y, t)$ is highly accurate. The relative error increases gradually with time, but remains very small, reaching 0.001042 at $t = 1$. This shows that $u_{HAM}(x, y, t)$ closely follows the exact solution over the full interval.

Looking at Table 3, the approximation for $v(x, y, t)$ has the largest relative error among the recovered quantities. The relative error increases steadily from 0 at $t = 0$ to 0.172321 at $t = 1$. However, this should be interpreted carefully because the exact values of v are small. Therefore, the relative error appears large even though the direct numerical difference between $v_{HAM}(x, y, t)$ and $v(x, y, t)$ remains moderate.

It can be seen that the moving boundary $s(t)$ is also well approximated in Table 5. The relative error remains below 0.038642 over the full time interval. The smallest nonzero relative error occurs at $t = 0.8$, while the largest occurs at $t = 1$. This indicates that the recovered boundary remains close to the exact boundary, with only mild deterioration at later time levels.

The pressure approximation $p_{HAM}(x, y, t)$shows a different pattern in Table 4. The relative error is largest at $t = 0.1$, where it reaches 0.099943. After that, the pressure error decreases significantly and becomes very small at $t = 0.5$, before increasing slightly again at $t = 1$. This suggests that the pressure recovery is most sensitive near the beginning of the simulation but remains stable overall.

Finally, the recovered coefficient $g_{HAM}(t)$ in Table 6 agrees with the exact coefficient $g(t)$ at all listed time levels. The relative error is zero throughout the table, showing that the overdetermined condition successfully recovers the unknown time-dependent coefficient.

The pressure recovery in Table 7 exhibits different error behavior for $\alpha = 0.95$. We see larger pressure errors at early times, with the maximum relative error occurring at $t = 0.1$. The relative error decreases steadily as time increases, and the approximation becomes more accurate at later time levels. In particular, at $t = 1$, the relative error for $\alpha = 0.95$ is 0.005703, compared with 0.033653 for $\alpha = 0.75$. Therefore, the smaller fractional order gives better early- and mid-time pressure recovery, while the larger fractional order improves the late-time pressure approximation.

6. Conclusion

In this study, the Homotopy Analysis Method was applied to a coupled two-dimensional inverse time-fractional Navier–Stokes moving-boundary problem involving a Caputo time-fractional derivative. The proposed formulation required the simultaneous recovery of the horizontal velocity $u(x, y, t)$, vertical velocity $v(x, y, t)$, pressure field $p(x, y, t)$, moving boundary $s(x, t)$, and unknown time-dependent coefficient $g(t)$. A manufactured exact solution was used to validate the semi-analytical approximation and to evaluate the accuracy of the recovered quantities.

A major feature of the present approach is the use of time-dependent convergence-control parameters $h_1(t)$ and $h_2(t)$, which were determined from prescribed boundary data rather than from the conventional use of *h-curves*. This boundary-driven selection provides a practical way to control the HAM approximation while remaining consistent with the physical data imposed on the problem. The second-order HAM truncation was found to give the most stable full-system approximation, while higher-order corrections increased computational instability, especially in the nonlinear pressure-sensitive terms. This supports the use of a low-order HAM approximation for the present coupled inverse moving-boundary model.

The numerical results for $\alpha = 0.75$ show that the HAM approximation accurately recovers the horizontal velocity, moving boundary, pressure field, and unknown coefficient. The approximation for $u_{HAM}(x, y, t)$ remains very close to the exact solution over the full time interval, with a maximum relative error of 0.001042. The moving boundary $s_{HAM}(x, t)$ also remains close to the exact boundary, with relative errors below 0.038642. The recovered coefficient $g_{HAM}(t)$ agrees exactly with the exact coefficient at all listed time levels, showing that the overdetermined condition successfully identifies the unknown time-dependent coefficient. The vertical velocity component has the largest relative error, but this is mainly due to the small magnitude of the exact values of $v(x, y, t)$; the direct deviation remains moderate.

The pressure recovery exhibits a more sensitive error pattern. For $\alpha = 0.75$, the pressure approximation is most sensitive near the beginning of the time interval, with the largest relative error occurring at $t = 0.1$. After this early-time region, the pressure error decreases significantly and remains stable over the rest of the interval. When the fractional order is increased to $\alpha = 0.95$, the pressure approximation displays larger early-time errors but improved late-time accuracy. In particular, at $t = 1$, the relative error for $\alpha = 0.95$ is smaller than that obtained for $\alpha = 0.75$. Thus, the results suggest that the fractional order influences the pressure recovery, with the smaller fractional order giving better early- and mid-time pressure behavior, while the larger fractional order improves the late-time pressure approximation.

Overall, the results demonstrate that the proposed HAM-based framework provides an effective semi-analytical approach for solving coupled inverse time-fractional Navier–Stokes moving-boundary problems. The method successfully recovers the velocity field, pressure, moving boundary, and unknown coefficient in a unified formulation. The study also shows that fractional-order sensitivity is important in pressure recovery and should be examined when validating time-fractional fluid-flow models. Future work may extend the present framework to other fractional operators, such as the Atangana–Baleanu–Caputo or Caputo–Fabrizio derivatives, and to more physically complex moving-boundary problems involving nonlinear interface conditions, variable viscosity, or non-manufactured data.

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Conflict of Interest Statement
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