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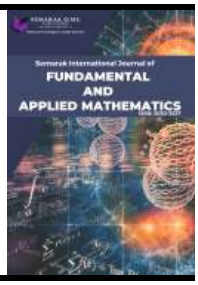

Benchmarking Numerical Computation of the First-order Polarization Tensor in Low Conductive Materials

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ABSTRACT

The study of the behaviour of conductive materials responding to electromagnetic fields is a key to the development in the field of materials science and engineering. Polarization Tensor (PT) provides the information regarding the shape, orientation and the material properties of an object. Current numerical techniques tend to be fast at the expense of accuracy leading to slowness to converge or expensive computation. This paper presents a simplified numerical approach, which will evaluate the first-order PT in three-dimensional, more efficiently and accurately. The discretization of the geometry is performed using free software which are Netgen Mesh Generator and MATLAB. Adaptive methods are used to shapes such as spheres and cubes, such as the selective mesh refinement are comparing 1-point and 3-point integration to achieve the optimum balance. The analytical solution of spherical geometry is used to validate proposed method. The use of adaptive mesh refinement with a matrix calculation method is much faster and more accurate in calculating PT. All these enhancements are on the whole a sensible step in the direction of simulating complex materials and forms a high-performance baseline on the real world applications such as medical imaging and industrial testing. The results show that the adaptive approach is a good and scalable method of computing the PT of complex conducting materials.

1. Introduction

The electromagnetic response of conductive materials is a topic of study in contemporary science and engineering, particularly in areas such as medical imaging, non-destructive testing, and composite materials design. Here, the Polarization Tensor (PT) is a simply defined mathematical object, which captures fundamental data regarding the shape, orientation and the material behaviour of a conducting inclusion. The mathematical basic of PT and moment tensors was developed systematically by Ammari and Kang [2] who applied it to the inverse problems such as electrical impedance tomography (EIT) and object detection and the effective medium theory of

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composite materials. Their research contains not only a rigorous proof but also a numerical algorithm to implement it, followed by the following computational results.

Only simple geometries, like these of a sphere and ellipsoid, do have analytic expressions in terms of the PT in this case. Recent papers have demonstrated that most of the available numerical methods tend to compromise accuracy in favour of speed, causing low convergence and instability especially in three-dimensional problems that were introduced by Amad *et al.,* [1] and Khairul Akmar *et al.,* [12].

In modern science and engineering, PT are used in medical imaging, detection as well as in material analysis. From research conducted by Sukri *et al.,* [17], the researcher demonstrate the numerical approach that has been proposed can reduce the computational complexity, while still maintaining accuracy.

In medical imaging application, PT can easily identify key characteristics of latent anomalies such as its location, size and conductivity by measuring the allocating along the boundary. This method, originally suggested by Ammari and Kang [4], and generalized by more recent studies Ammari *et al.,* [1], significantly reduces the computational cost, though with the same amount of information retained. Moreover, there are researcher that expanded the use of generalized polarization tensors (GPTs) which have higher-order geometric content and anisotropic characteristics of objects,for example research by Ammari *et al.,* [1] shows that frequency dependent PT can be used to recover the same shapes such as ellipses, and internal electromagnetic parameters, which demonstrates that the last one are also practical in detection and quantitative description. PT has been shown to have great practical value in nonlinear inverse problems, especially in the process of reconstructing small well-separated inclusions. Watson *et al.,* [19] showed that PT allows fast and robust reconstruction by reconstructing nonlinear saturation using asymptotic expansion. It provides computationally inexpensive approximation of the Hessian term to speed up the implementation of electrical impedance tomography.

PT are also of importance in materials science, especially in the analysis of composite materials and the homogenization theory. They occur as expressions to combine the effective macroscopic properties of a material with the properties and geometry of the material microscopic inclusions. To generalize composites when a composite is dilute, the total conductivity or permittivity can be sorted into PT of individual inclusions as also noted in both classical and more recent numerical models to generalize studies by Amad *et al.,* [1] and Khairul Akmar *et al.,* [12]. In analyzing materials, PT can be effectively used to model how changes in shape, size, and orientation of inclusions modulate the overall behaviour of the material, without necessarily solving any detailed micro-scale problems. This has been particularly useful in modelling more advanced materials like metamaterials, where the microscale polarization response determines the macroscale response as discussed in Zhang *et al.,* [20] and Intaravanne *et al.,* [9].

In conclusion, the PT are important to guide the theoretical modelling and practical application. The use of them in imaging and materials science and many other fields is very useful due to their capability to simplify the model of complex wave interactions, to reduce the costs of computation and improve stability during an inverse problem. PT has become increasingly important in modern research due to continuous improvements in theory and numerical methods, and they are expected to play an even bigger role in future technological developments. This paper aims to evaluate first order PT for geometries with analytical solution provided by Ammari and Kang [2]. The technique proposed is based on the Gaussian quadrature which will be used to evaluate the main integral of PT. The technique is based on mesh discretization via Netgen Mesh Generator and numerical integration of PT integral which will be done using MATLAB software with an emphasis on increasing number of surface elements and matrix-based computation.

1.1 Challenges in Numerical Computation of PT

Accurate numerical computation of PT are important in dealing with engineering application, such as electromagnetic imaging, modelling of composite materials, and inverse conductivity problems. Tensors are useful because of its ability to represent the response of small inputs to external fields. Previous research on the numerical approach to solve for PT has led to efficient algorithms, as introduced by Sukri *et al.,* [18], which highlight their importance in practical computations. Nonetheless, even with recent advances in the last few years, a number of issues are still resolved to come up with accurate, stable and efficient numerical estimates.

By Khairul Akmar *et al.,* [12] demonstrated that the density of discretization points has a significant effect on the precision of the calculated tensor, especially when dealing with highly conductive objects. The benchmark study presented by Amad *et al.,* [1] highlights the importance of mesh discretization and proper quadrature schemes to control these numerical errors.

Another important problem is associated with material contrast and numerical conditioning. The more dissimilar the separation between the inclusion and the surrounding medium, the worse the equations governing it. This can lead to instabilities in numerical solvers such as oscillatory solutions or slow convergence. Computational of PT has become sensitive to contrast as introduced by Schneider [16] because the researcher shows that the type of polarization computational method exploits that the convergence of such methods strongly depends on the choice of reference material used in FFT-based computational micromechanics. This shows that the PT is a physical descriptor not only because it is closely related to the numerical schemes that approximate it.

The higher-order GPTs is computationally more complex. Higher-order terms are needed to obtain better resolution and more accurate reconstruction in many inversion and imaging applications. But these terms are much more sensitive to discretization errors and numerical noise. This growth of higher order coefficients in terms of their algebraic sensitivity has already been observed by Capdeboscq *et al.,* [7]. An example of this can be provided by the results presented by Amad *et al.,* [1] whereby the precision of higher-rank tensors is compromised whenever the resolution of the layer of the boundaries is inefficient. Similarly, recent developments by Khairul Akmar *et al.,* [12] emphasise on precision in avoiding errors when applying interpolation method to compute PT.

In summary, although significant progress has been made in numerical calculations of PT, there are still challenges to be overcome. The problems that still affect the quality and stability of the results are mesh sensitivity, numerical conditioning, and high-order tensors. Recent studies are trying to develop better, more effective and strong processes to counter these inefficiencies. Since PT are gaining popularity in a number of applications in science and engineering today, there is a need to know how to calculate them as numbers to solve practical problems.

2. Methodology

2.1 Mathematical Formulation of the PT

The mathematical formulation of the first-order PT is derived from the transmission problem and asymptotic expansion of the perturbed field. As introduced by Ammari and Kang [2], the PT can be expressed as a boundary integral over the surface of a conducting inclusion. Based on the formulation The integral of PT, M_{ij} is defined as equation below :

$$M_{ij} = \int_{\partial B} Y^j \phi_I (Y)\, d\sigma(Y)\,, \quad Y \in \partial B, \tag{1}$$

where $\phi_i$is the solution of the associated boundary integral equation and $Y \in \partial B$represents a point on the boundary.
From Eq. (1), the unknown density function $\phi_i$is obtained by solving the following linear system of equation:

$$\emptyset_i(Y) = (\lambda I - K_B^*)^{-1}(V_x.\square X^i)(Y), \quad (2)$$

where the parameter λ, which depends on the conductivity contrast, is defined as in Eq (3):

$$\lambda = \frac{k+1}{2(k-1)}, \quad (3)$$

The operator K_B^*, known as the Neumann–Poincaré operator, is a singular integral operator given by:

$$K_B^*[\phi](X) = \frac{1}{4\pi}\ P.V. \int_{\partial B} \frac{\langle X-Y\rangle\langle V_X\rangle}{|X-Y|^3}\ \phi(Y)\ d\sigma(Y). \quad (4)$$

From Eq. (1)– (4), these three equations form the fundamental framework of PT computation, where the operator is first constructed, followed by solving the boundary density function, and finally computing the polarization tensor. In this research we will used interpolation method which are linear and quadratic element interpolation. By referring to the research by Sukri *et al.,* [18], the researcher uses quadratic element interpolation in order to calculate the first order PT. Hence, in this study, we are going to re-evaluated the first order PT for sphere as in study by Sukri *et al.,* [18] and continue to computed the first order PT for other geometry which is ellipsoid. The additional part in this paper is, we will do the comparison between the numerical approach by Sukri *et al.,* [18] and linear element integration by Khairuddin *et al.,* [11].

2.2 Analytical Solution for Benchmark Geometries

To validate the numerical computation of the PT, analytical solutions for simple geometries such as sphere and ellipsoid are employed for which:
for a spherical domain of radius a, PT is isotropic and is given by:

$$M = 4\pi a^3 \frac{k-1}{k+2} I, \quad (5)$$

where I is the identity matrix. This indicates that all diagonal components are equal, reflecting the symmetry of the geometry.

For an ellipsoidal domain with semi-axes a_1, a_2, a_3, the PT is diagonal but anisotropic, and can be expressed as below:

$$M_{ii} = |\ B\ | \frac{k-1}{1+(k-1)L_i}, i = 1,2,3 \quad (6)$$

where $|\ B\ | = \frac{4}{3}\pi a_1 a_2 a_3$ represents the volume of the ellipsoid, and $L_i$are the depolarization factors satisfying:

$$L_1 + L_2 + L_3 = 1. \quad (7)$$

In contrast to the spherical case, the diagonal components $M_{ii}$are not identical, reflecting the directional dependence of the ellipsoidal geometry.

These analytical solutions are employed as reference benchmarks to verify the accuracy and reliability of the numerical results.

3. Results

3.1 Validation and Convergence Analysis (Sphere)

We begin the computation of first order PT by computing the PT for sphere with a radius of $r = 0.01$ and a conductivity contrast parameter of $k = 1.5$. This domain acts as a standard sphere since it is known to have an analytical solution and provides the means of directly verifying the numerical methods. Netgen discretizes the sphere with triangular surface meshes, the number of elements on the surface, N_{mesh} analyzing in resolutions 118, 176, 230, 288, 472, 620, 704, 920, 1152 and 1888 to systematically examine convergence and relative error of both linear element and quadratic element integrations.

The graph clearly shows the convergence characteristics in the spherical case. As shown in Figure 1 below, as number of elements on the surface, the more the diagonal values are closer to the exact reference line, then the numerical model converges as the mesh is refined. In the case of quadratic elements, the values are very close to the exact answer and only small variations with mesh narrowing are observed. The linear elements are further from the exact value, but they improve progressively as the surface mesh increases and end up at the same limit. In general, this value is confirmation that lateral discretization produces better results and that the quadratic approximation is more efficient and fits the sphere. This process is uniform and it can be used to calculate the PT accurately.

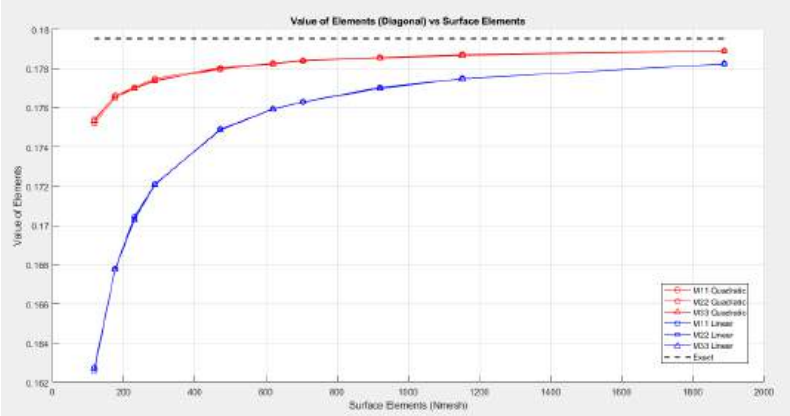

Fig. 1. Value of elements (diagonal) vs surface elements

The symmetry and stability of the proposed numerical framework is significantly confirmed in the comparison of the off-diagonal components M_{13} and M_{31} of the spherical domain. These components should be zero in the case of a perfect spherical analysis since they are geometrically symmetric. Based on the findings as shown in Figure 2 below, both the linear and quadratic terms converge to zero as the number of surface elements increases, indicating that the linear and quadratic terms are numerically stable. This overlapping behaviour M_{13} and M_{31} is another indication that the reciprocal property $M_{ij} = M_{ji}$ is always maintained during the calculations which is a sign of correct physical formulation. Linear elements have larger initial fluctuations and require a reduced mesh to approach zero as far as performance is concerned. Quadratic elements on the other hand can converge more smoothly and more effectively to an error bound close to zero making them more effective in precisely resolving curved geometries. Overall, the findings show that the adaptive numerical method does not remove numerical artifacts completely, but it does not affect tensor

symmetry, implying that the method used to calculate PT provides accurate and reliable values when the mesh is fined.

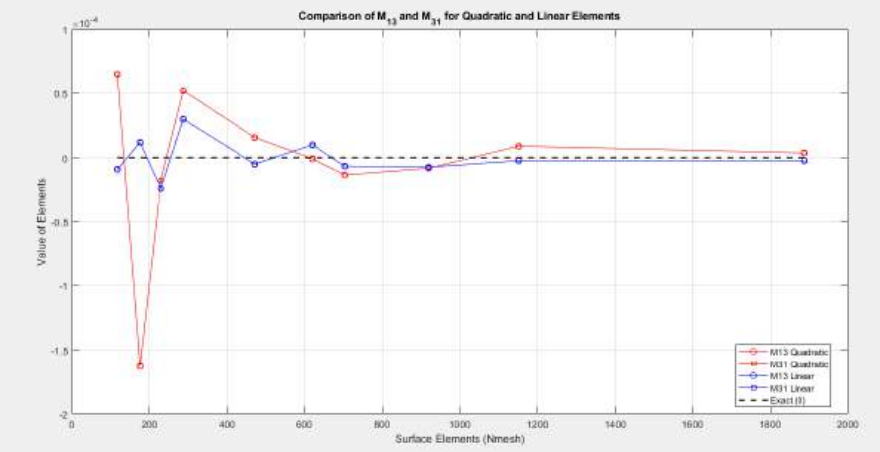

Fig. 2. Comparison of M_{13} and M_{31} for quadratic and linear elements

In Figure 3 below, the accuracy and symmetry of the proposed numerical method is also proven by comparing the off-diagonal components M_{12} and M_{21} in the case of a spherical domain. The components, in the case of a perfect sphere, should be zero due to geometric symmetry. The results show that both components gradually approach zero as the number of surface elements increases, confirming the symmetric interaction property $M_{ij} = M_{ji}$ and the physical consistency of the model. The fact that M_{12} and M_{21} are close means that the numerical scheme does not introduce asymmetries when comparing itself to the numerical scheme. Regarding the convergence behaviour, linear elements have larger initial deviations and need to be penalized to obtain a closer representation of zero. Quadratic elements, on the other hand, have a faster and smoother convergence, approaching values close to zero with much fewer surface elements, due to their greater efficiency in describing curved geometries. In general, the findings support the idea that the numerical framework is practical in removing off-diagonal numerical noise because the mesh is smoothly refined, and quadratic integration is more accurate and stable in the calculation of the PT in symmetric geometry.

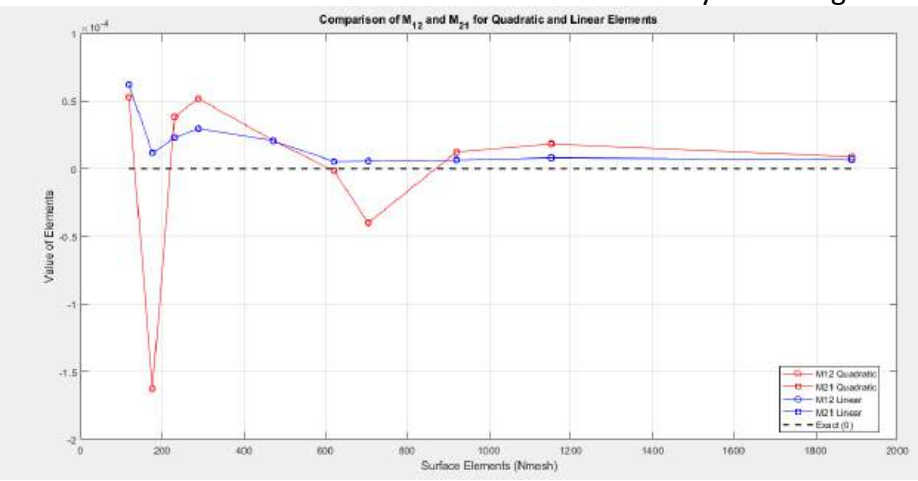

Fig. 3. Comparison of M_{12} and M_{21} for quadratic and linear elements

The mathematical consistency of the proposed framework is further confirmed by the fact that the off-diagonal elements M_{23} and M_{32} are verified to the spherical domain as shown in Figure 4

below. In the case of a perfectly symmetric sphere, the term should be zero due to geometric symmetry. It is found that both elements approach zero smoothly as the number of surface elements increases and this indicates that the correct tensor reciprocal $M_{ij} = M_{ji}$ is being used and there is no directional bias in the numerical model. The fact that M_{23} and M_{32} are almost the same at all levels of discretization further proves the stability of the implementation. Linear elements are more volatile with lower mesh density in convergence behavior and require finer discretization to be accurate. In contrast, quadratic elements have a faster and smoother decay to zero, which indicates their ability to reduce geometric approximation errors on curved surfaces. In general, the study concludes that the adaptive numerical method is sound and robust and quadratic elements provide better performance and accuracy for high-accuracy PT calculations.

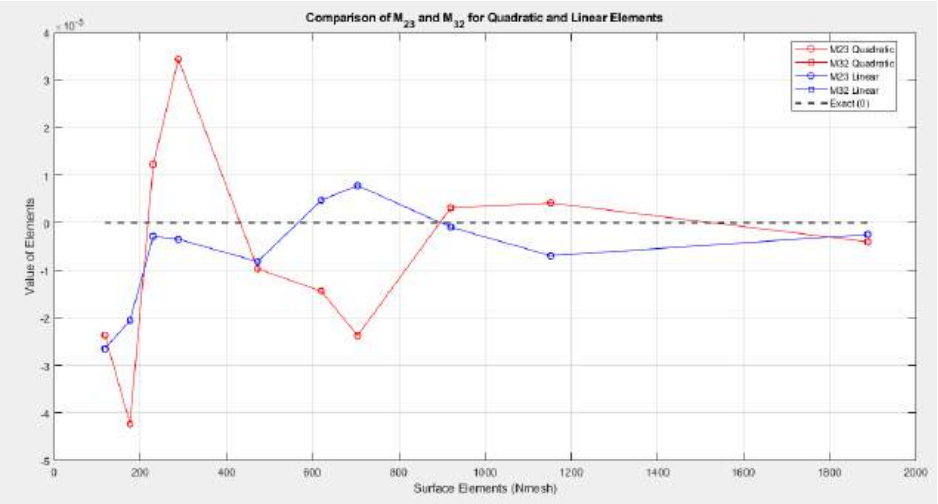

Fig. 4. Comparison of M_{23} and M_{32} for quadratic and linear elements

In Figure 5 below shows the graph validates the proposed numerical structure by comparing the obtained PT and the analytical solution for the spherical unit. The findings generally show a significant improvement in accuracy as the mesh is successively finer, with a very high convergence behaviour. The reduction in relative error decreases as the number of surface elements increases, proving that the method is numerically stable and leads to an accurate answer. In particular, the linear element scheme exhibits a slow error decrease even though it requires a high mesh density to be quite accurate. In contrast, the quadratic element formulation can be seen to achieve a much lower error with almost half of the elements showing a high level of efficiency in representing the curved geometry of the sphere. The relative error decreases below 1500 surface elements. 10^{-3} determines that the computational structure is at a high level of accuracy. This finding indicates that the method used is reliable in performing reliable PT evaluations when implementing material issues. In general, it can be observed that the adaptive integration approach is a suitable strategy for eliminating numerical errors as well as improving computational efficiency and applications in high-precision simulations in materials science.

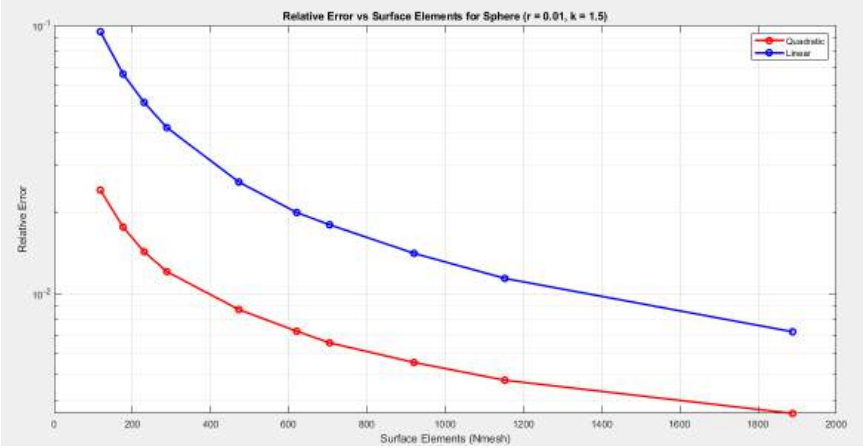

Fig. 5. Relative error vs surface elements for sphere

3.2 Numerical Performance of PT Ellipsoid on Irregular Geometries (Ellipsoid)

Using ellipsoidal inclusions as the test geometry, we begin the computation of the first-order PT. The ellipsoid is defined by the semi-principal axes, a=2, b=1, and c=1, which give it a prolate shape extended down the x-axis. The conductivity of the inclusion is set to $k = 0.1$, while the surrounding medium is assumed to have unit conductivity. It is selected based on an exact PT expression that can be used directly to verify the numerical results. Table 1 shows the computed PT for an ellipsoidal geometry with conductivity k = 0.1, using both linear and quadratic elements at different mesh densities.

Table 1
Comparison of PT for linear and quadratic elements

Nmesh, N	Linear Element	Quadratic Element
170	$\begin{bmatrix} -8.28392132 & 0.00552830 & 0.00443287 \\ -0.00218579 & -11.13277723 & -0.01945985 \\ -0.00146579 & -0.01934901 & -11.15242639 \end{bmatrix}$	$\begin{bmatrix} -8.84237739 & 0.02358905 & 0.04669449 \\ -0.01315557 & -11.8778188 & -0.02737037 \\ 0.02051638 & -0.03325946 & -11.84661642 \end{bmatrix}$
444	$\begin{bmatrix} -8.84237739 & 0.02358905 & 0.04669449 \\ -0.01315557 & -11.8778188 & -0.02737037 \\ 0.02051638 & -0.03325946 & -11.84661642 \end{bmatrix}$	$\begin{bmatrix} -8.99697743 & -0.00531659 & -0.01739977 \\ -0.01097191 & -12.03419355 & -0.00367713 \\ 0.000754208 & 0.00018606 & -12.02849885 \end{bmatrix}$
668	$\begin{bmatrix} -8.84237739 & 0.02358905 & 0.04669449 \\ -0.01315557 & -11.8778188 & -0.02737037 \\ 0.02051638 & -0.03325946 & -11.84661642 \end{bmatrix}$	$\begin{bmatrix} -8.98512439 & -0.00116415 & -0.01339304 \\ 0.00259728 & -12.04517693 & 0.00345249 \\ 0.00154009 & 0.00409757 & -12.04973719 \end{bmatrix}$

The calculation of PT of the ellipsoids geometry was evaluated by considering linear and quadratic elements with various mesh densities and matched with the first-order analytical PT. Analytical solution shows that the diagonal elements have value -1.11720250, -1.11720250 and -1.60238593 implying that the ellipsoid exhibits anisotropic behaviour. The numerical findings show convergence towards the analytical solution as the mesh density increases from N=170 to N=668. Specifically, the diagonal elements M_{11}, M_{22}, M_{33}, and produced with quadratic elements have a steady rise in magnitude that gets closer to the precise values. Although they converge more slowly, the linear elements generally show a similar pattern.

Even though the numerical values are still smaller when compared to the analytical PT, the tendency of steadily moving towards the correct solution proves the legitimacy of the proposed method. The permutation of the diagonal elements. The ordering of the diagonal elements $M_{33} > M_{22} > M_{11}$ is maintained in all the numerical cases also, shows the anisotropy of the ellipsoidal geometry, which is consistent with the analytical result. Moreover, the off-diagonal terms are also relatively minor and stabilize with mesh refinement, especially in the quadratic elements. This shows enhanced numerical stability and regularity of the technique.

In general, the quadratic element formulation offers improved results in comparison to linear element in terms of agreement with the analytical representation and superior accuracy and convergence behaviour towards the ellipsoidal case.

3.3 Error Analysis for Ellipsoidal Geometry

An error analysis is used to check how accurate the proposed numerical framework is by comparing the relative error of the computed PT from both linear and quadratic elements to the analytical first-order solution. The precise PT values serve as the benchmark solution for error evaluation. Table 2 presents the corresponding error results for various mesh densities.

Table 2
Relative error of PT (ellipsoidal geometry)

Nmesh	Error Analysis	
	Linear Element	Quadratic Element
170	0.07290379	0.01043685
444	0.02165145	0.00686561
668	0.01434766	0.00553673

According to Table 2, the relative error analysis of PT for ellipsoidal geometry is provided under linear and quadratic elements after varying mesh densities. The relative error for the linear element decreases dramatically from 0.07290379 at N = 170 to 0.02165145 at N = 444 and then to 0.01434766 at N = 668. Although the convergence rate levels off at larger mesh densities, this consistent decrease demonstrates that mesh refinement is an efficient method of determining PT. This indicates that there is an underlying restriction of the first-order approximation in explaining the geometric consequences of high-order, and that as the linear component refines, the accuracy improvement of the component progressively becomes saturated.

Comparatively, the quadratic element has a more uniform value of relative errors at all mesh sizes. The error decreases from 0.01043685 at N = 170 to 0.00686561 at N = 444, and further to 0.00553673 at N = 668. The quadratic element results are much more accurate over all though the reduction rate is a more gradual process than in the linear case. This confirms the higher order of approximation in elements of a higher order able to approximate the geometry and field variation more accurately even at coarse meshes. Relative to the linear element, the quadratic element interpolation method has an order of magnitude reduced error at all mesh levels. As an example, the quadratic element error is approximately seven times less at N = 170 than it is at N = linear element, though at N = 668, it is almost three times more exact. It shows that higher-order discretisation not only yields greater accuracy, but also better performance using fewer elements. Finally, it can be concluded that the results are quite clear that both approaches approach the actual solution when the mesh is refined, but the quadratic element component consistently better than the linear element, in accuracy and efficiency. The convergence behaviour observed confirms the efficiency of the higher-order elements in PT calculation, especially regarding high accuracy at a low cost of computation.

4. Conclusions

In conclusion, this paper has fulfilled its objective of creating an adaptive numerical framework for the correct calculation of the PT. The results prove that the presented method involving a combination of linear and quadratic surface discretization can provide a technique with the ability to handle geometries of different complexity with a higher level of reliability. In particular, the quadratic formulation has been shown to produce more accurate results than the linear method compared to the exact analytical solution for both spherical and ellipsoidal domain.

The findings also show that the choice of integration system and geometric representation is crucial in obtaining the general accuracy and stability of the calculation in PT. Although the numerical results exhibit good consistency with the theoretical values, the small discrepancies of other coarser meshes and complicated geometries indicate that the method is sensitive to the quality of the mesh and the level of discretization. Therefore, it is suggested to refine it further and perform more numerical experiments to fully certify the robustness of the proposed framework in more challenging conditions.

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