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Generation of Picture Fuzzy B-Spline Curve Interpolation with Open Uniform Knot Vector

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ABSTRACT

An open uniform knot vector ensures that the resulting B-spline curve starts at the first control point and ends at the last control points, providing a more intuitive and predictable shape at the boundaries. When dealing uncertainty in data, open uniform knot vector B-spline are particularly useful because they offer several advantages. In this paper, the picture fuzzy set approach was used to introduce the picture fuzzy B-spline curve interpolation model with open uniform knot vector. Firstly, picture fuzzy control point relation is introduced by using basic concepts of picture fuzzy set which is involved picture fuzzy number and picture fuzzy relation. The B-spline basis function is then blended with open uniform knot vector with order $k=4$ and the picture fuzzy control point relation. Afterwards, using interpolation technique, fuzzy B-spline curves interpolation are generated and visualized using open uniform knot vector. In conclusion, a few numerical examples for creating the desired curve are displayed.

1. Introduction

The open uniform knot vector is a powerful tool in B-spline theory, providing a balance between flexibility and control, making it a popular choice in various fields requiring precise curve and surface representation curve. Specifically, an open uniform knot vector ensures that the first control point is where the B-spline curve begins, and the last control point is where it ends, making it common choice in many applications for its desirable properties. Research on B-spline and their applications, including the development and utilization of open uniform knot vectors, has been a significant area within computational mathematics, computer graphic and CAGD [1,2]. Yeh *et al.,* [3,4] introduced a method using high-order derivatives for knot placement, which enhances the accuracy of B-spline approximation by ensuring equal approximation error throughout the domain. In 2020, Michel and Zidna [5] focused on deterministic heuristic knots placement for B-spline curve approximation, proposing adaptive algorithms that improve the handling of dense and noisy data points. There are several researchers who have studied B-spline interpolation with open knot vectors on various field

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for curve and surface [6,7]. However, B-spline curves interpolation assume precise and deterministic data, which is often not the case in real-world applications. Data can be imprecise, uncertain, or incomplete due to various factors such as measurement errors, noise or inherent variability in the phenomena being modelled.

Zadeh in 1965 [8] developed the concept of the fuzzy set to address the limitations of classical set theory in dealing with uncertainty and imprecision. The fuzzy set concept allows for degree of membership, meaning an element can partially belong to a set to varying degrees between 0 and 1. However, there is a weakness in this fuzzy set concept where it cannot handle more complex and nuanced situations of uncertainty and vagueness. Coung and Kreinovich [9,10] introduced the picture fuzzy set (PFS) as an expansion of the fuzzy set (FS) and intuitionistic fuzzy sets (IFS) [11]. PFS is designed to handle more nuanced and complex situations of uncertainty, incorporating additional parameters to represent neutrality or hesitancy, along with degrees for membership and non-membership. To further their investigation, Coung [10] has included the notions of positive, neutral, and negative membership degrees for an element that is an extension of an IFS. After PFS was developed, it was thought to be a powerful mathematical tool that worked well in scenarios where human opinions involved a greater number of yes, abstain, no, and refuse responses. A few studies on the generalizations of the picture fuzzy set idea in various disciplines and its application to a problem involving decision-making have been carried out. A geometric explanation of picture fuzzy sets was given by Singh [12], who additionally proposed the values of correlation for picture fuzzy sets that take into consideration the degree of positive, negative, neutral, and refuse membership. On the other hand, Wei [13], published similarity measures between picture fuzzy sets by taking into consideration the degree of positive, neutral, negative, and refusal membership in picture fuzzy sets. In 2018, Wei and Gao [14], introduced a few new dice measurement of similarity metrics for picture fuzzy sets as well as the generalized dice similarity metrics for picture fuzzy sets. In 2018, Wei [15], introduced a new method for calculating the degree of similarity between PFS. The author used these PFS similarity metrics for construction material and mineral field recognition. Consequently, a few applications utilizing PFS have been discussed by some academics in their study [16-20].

Picture fuzzy B-spline curve interpolation with open knot vectors is a method that integrates fuzzy logic with B-spline interpolation technique to handle data that is imprecise or uncertainty. Many researchers use the FS and IFS approaches for geometric modeling in B-splines [21-26]. However, Rosli and Zulkifly [27], presented the neutrosophic B-spline curve by employing the interpolation method and neutrosophic bicubic B-spline surface interpolation [28]. Meanwhile, Smarandache [29], has contributed significantly to picture fuzzy sets in various fields including geometric modelling. Additional researchers in the field who have utilized PF sets in various scenarios like image processing, pattern recognition, and decision making. These researchers have explored and extended the application of picture fuzzy sets to enhance the accuracy and flexibility of geometric modeling and other computational techniques [30,31].

The objective in this paper is to focus on the designing of geometric model to handle gathering uncertainty-related data which is focused on picture fuzzy B-spline curve interpolation (PFBSCI) model with open knot vectors. Before constructing the interpolation of PFBSC, the picture fuzzy control point needs to be established using PFS theories and their characteristics. PFBSC interpolation with open knot vectors models are created utilizing these control points and the B-spline basis function. The technique of interpolation is then utilized to view the models. The arrangement of this paper is as follows: Section 1 gives an introduction on this topic. Section 2 is methodology of this study which introduces the picture fuzzy control point relation (PFCPR) and picture fuzzy point relation (PFPR). Then, section 3 covered the technique for the PFBSC interpolation using PFCPR. The

PFBSC interpolation is shown graphically and numerically in section 4. Lastly, the research will be concluded in section 5.

2. Preliminaries

There are four types of membership functions in picture fuzzy set where positive, neutral, negative and refusal membership function denoted as μ,η,v and ρ.

2.1 Picture Fuzzy Point Relations

The concept of relation is one of the most important concepts in modeling. Some preliminary picture fuzzy relation (PFR) results are shown in [9,10]. Defining the picture fuzzy point relation (PFPR) based on PFS is explained in definition 1.

Definition 1
Let P and Q be collection of points non-empty set and $P,Q,I \subseteq {}^{\sim 3}$, PFPR is thus defined as

$$\hat{S}=\left\{\begin{array}{l}\left\langle\left(p_i,q_j\right),\mu_{\hat{S}}\left(p_i,q_j\right),\eta_{\hat{S}}\left(p_i,q_j\right),v_{\hat{S}}\left(p_i,q_j\right),\rho_{\hat{S}}\left(p_i,q_j\right)\right\rangle \mid \\ \left(p_i,q_j\right),\mu_{\hat{S}}\left(p_i,q_j\right),\eta_{\hat{S}}\left(p_i,q_j\right),v_{\hat{S}}\left(p_i,q_j\right),\rho_{\hat{S}}\left(p_i,q_j\right)\in I\end{array}\right\} \tag{1}$$

where $\left(p_i,q_j\right)$ are point in ordered pair and $\left(p_i,q_j\right)\in P\times Q$. While $\mu_{\hat{S}}\left(p_i,q_j\right)$, $\eta_{\hat{S}}\left(p_i,q_j\right)$, $v_{\hat{S}}\left(p_i,q_j\right)$ and $\rho_{\hat{S}}\left(p_i,q_j\right)$ are the classifications of positive, neutral, negative and refusal membership of the corresponding ordered pair of points in $\left[0,1\right]\in I$. The degree of refusal is indicated by

$$\rho_{\hat{S}}\left(x\right)=1-\left[\mu_{\hat{S}}\left(p_i,q_j\right)+\eta_{\hat{S}}\left(p_i,q_j\right)+v_{\hat{S}}\left(p_i,q_j\right)\right] \tag{2}$$

and the condition of $0\le\mu_{\hat{S}}\left(p_i,q_j\right)+\eta_{\hat{S}}\left(p_i,q_j\right)+v_{\hat{S}}\left(p_i,q_j\right)\le 1$ is followed.

Definition 2 [9]
In relation to a fixed $x\in\hat{A}$, $\left(\mu_{\hat{A}}\left(x\right),\eta_{\hat{A}}\left(x\right),v_{\hat{A}}\left(x\right),\rho_{\hat{A}}\left(x\right)\right)$ is call picture fuzzy number (PFN) with $\mu_{\hat{A}}\left(x\right)\in\left[0,1\right],\eta_{\hat{A}}\left(x\right)\in\left[0,1\right],v_{\hat{A}}\left(x\right)\in\left[0,1\right]\,\rho_{\hat{A}}\left(x\right)\in\left[0,1\right]$

And

$$\mu_{\hat{A}}\left(x\right)+\eta_{\hat{A}}\left(x\right)+v_{\hat{A}}\left(x\right)+\rho_{\hat{A}}\left(x\right)=1 \tag{3}$$

PFN is simply represented as $\left(\mu_{\hat{A}}\left(x\right),\eta_{\hat{A}}\left(x\right),v_{\hat{A}}\left(x\right)\right)$.

Definition 3 [9]
Let X, Y and Z be representations of normal non-empty sets. A picture fuzzy relation (PFR), $\hat{R}$ is a subset of picture fuzzy, $X\times Y$ defined by

$$\widehat{R}=\left\{\left((x,y),\mu_{\widehat{R}}(x,y),\eta_{\widehat{R}}(x,y),v_{\widehat{R}}(x,y)\right):x\in X,y\in Y\right\} \tag{4}$$

In which $\mu_{\widehat{R}}:X\times Y\to[0,1]$, $\eta_{\widehat{R}}:X\times Y\to[0,1]$ and $v_{\widehat{R}}:X\times Y\to[0,1]$ satisfied the condition $0\le\mu_{\widehat{R}}(x,y)+\eta_{\widehat{R}}(x,y)+v_{\widehat{R}}(x,y)\le 1$ for every $(x,y)\in(X,Y)$

The set of all picture's fuzzy relation in $X\times Y$ will be denoted by $(X\times Y)$.

2.3 Picture Fuzzy Control Point Relation

Control points are the grouping of all sets of points that are utilized to ascertain a spline curve's shape. The process of creating, regulating, and constructing a smooth curve involves the control point. The fuzzy control point notion from earlier research [32][33] is used to visualize PFCPR. PFCPR is defined as follows:

Definition 4
Assuming $\widehat{S}$ to be a PFPR, PFCPR can be defined as a group of points $n+1$ that are used to characterize the curve and are located and represented by coordinates as

$$\begin{aligned}\widehat{C}_i^{\mu}&=\left\{\widehat{C}_0^{\mu},\widehat{C}_1^{\mu},\widehat{C}_2^{\mu},..,\widehat{C}_{n+1}^{\mu}\right\}\\ \widehat{C}_i^{\eta}&=\left\{\widehat{C}_0^{\eta},\widehat{C}_1^{\eta},\widehat{C}_2^{\eta},..,\widehat{C}_{n+1}^{\eta}\right\}\\ \widehat{C}_i^{v}&=\left\{\widehat{C}_0^{v},\widehat{C}_1^{v},\widehat{C}_2^{v},..,\widehat{C}_{n+1}^{v}\right\}\\ \widehat{C}_i^{\rho}&=\left\{\widehat{C}_0^{\rho},\widehat{C}_1^{\rho},\widehat{C}_2^{\rho},..,\widehat{C}_{n+1}^{\rho}\right\}\end{aligned} \tag{5}$$

Where $\widehat{C}_i^{\mu},\widehat{C}_i^{\eta},\widehat{C}_i^{v}$ and $\widehat{C}_i^{\rho}$ are PFCP for positive, neutral, negative and refusal membership where i is one smaller than n (the number of points).

2.4 Picture Fuzzy B-spline Curve Interpolation

The PFBSCI is defined and produced by combining the PFCPR with basis function of B-spline.

Definition 5
Let $\widehat{C}_i=\widehat{C}_i^{\mu},\widehat{C}_i^{\eta},\widehat{C}_i^{v},\widehat{C}_i^{\rho}$ where $i=1,2,3,..,n+1$ be a PFCPR and PFBSCI represented by $\widehat{S}(t)$ using the location vector across the curve as a function of the parameter t, PFBSCI is expressed as

$$\widehat{S}(t)=\sum_{i=1}^{n+1}\widehat{C}_i N_i^k(t) \tag{6}$$

With $t_{\min}\le t\le t_{\max}$ and $2\le k\le n+1$ where $\widehat{C}_i$ are the position vector of $n+1$ control polygons vertices and the N_i^k are the B-spline basis function where its described as

$$N_i^1(t) = \begin{cases} 1 & if \;\; t_i \le t < t_{i+1} \\ 0 & otherwise \end{cases} \tag{7}$$

And

$$N_i^k(t) = \frac{(t - t_i) N_i^{k-1}(t)}{t_{i+k-1} - t_i} + \frac{(t_{i+k} - t) N_{i+1}^{k-1}(t)}{t_{i+k} - t_{i+1}} \tag{8}$$

PFBSC in Eq. (6) is parametric function consists of positive, neutral, negative and refusal membership curve described as,

$$\widehat{S}_\mu(t) = \sum_{i=1}^{n+1} \widehat{C}_i^\mu N_i^k(t) \tag{9}$$

$$\widehat{S}_\eta(t) = \sum_{i=1}^{n+1} \widehat{C}_i^\eta N_i^k(t) \tag{10}$$

$$\widehat{S}_v(t) = \sum_{i=1}^{n+1} \widehat{C}_i^v N_i^k(t) \tag{11}$$

$$\widehat{S}_\rho(t) = \sum_{i=1}^{n+1} \widehat{C}_i^\rho N_i^k(t) \tag{12}$$

The data point should be in Eq. (6) if the data points are within the PFBSCI range. For each data point indicated by W_j, Eq. (6) has been modified as follows:

$$\begin{aligned} &\widehat{D}_1(t_1) = W_1^k(t_1)\widehat{B}_1 + W_2^k(t_1)\widehat{B}_2 + \ldots + W_{n+1}^k(t_1)\widehat{B}_{n+1} \\ &\widehat{D}_2(t_2) = W_1^k(t_2)\widehat{B}_1 + W_2^k(t_2)\widehat{B}_2 + \ldots + W_{n+1}^k(t_2)\widehat{B}_{n+1} \\ &: \\ &\widehat{D}_j(t_j) = W_1^k(t_j)\widehat{B}_1 + W_2^k(t_1)\widehat{B}_2 + \ldots + W_{n+1}^k(t_j)\widehat{B}_{n+1} \end{aligned} \tag{13}$$

When $2 \le k \le n+1 \le j$. Eq. (13) can be express as a matrix,

$$\left[\widehat{D}\right] = [W]\left[\widehat{B}\right] \tag{14}$$

$$\begin{aligned} &\left[\widehat{D}\right]^T = \left[\widehat{D}_1(t_1) \quad \widehat{D}_2(t_2) \quad \ldots \quad \widehat{D}_j(t_j)\right] \\ &[W] = \begin{bmatrix} W_1^k(t_1) & \cdots & \cdots & W_{n+1}^k(t_1) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ W_1^k(t_j) & \cdots & \cdots & W_{n+1}^k(t_j) \end{bmatrix} \\ &\left[\widehat{B}\right]^T = \left[\widehat{B}_1 \quad \widehat{B}_2 \quad \cdots \quad \widehat{B}_j\right] \end{aligned} \tag{15}$$

The measurement of data points along PFBSCI is the value of metrix t_j for each result. The parametric value on data point to l for point of data point is j as follows.

$$t_1 = 0$$

$$\frac{t_l}{t_{\max}} = \frac{\sum_{r=2}^{l} \left| \widehat{D}_r - \widehat{D}_{r-1} \right|}{\sum_{r=2}^{j} \left| \widehat{D}_r - \widehat{D}_{r-1} \right|} \text{ for } l \geq 2 \tag{16}$$

The symbol $t_{\max}$ represents the maximum parameter for knot vector. If $2 \leq k \leq n+1 = j$, then the squared matrix W and control polygon can be immediately derived by using an inverse matrix as below.

$$\left[\widehat{B}\right] = \left[W\right]^{-1}\left[\widehat{D}\right], 2 \leq k \leq n+1 = j \tag{17}$$

As a result, PFBSCI can be acquired using Eq. (17).

3. Open Uniform Knot Vector

Eq. (7) and Eq. (8) makes it clearly apparent that the B-spline basis functions $N_i^k(t)$ are significantly impacted by the knot vector selection and depending on the ensuing B-spline curve. A knot vector can only exist if it satisfies the relation $x_i \leq x_{i+1}$ [2]. An open uniform knot vector's multiplicity of knot values at its endpoints is equal to the basis function of B-spline order k. Several instances of integer increment in use are $k = 2$, $k = 3$ and $k = 3$ or for standardized increases

$$k = 2 \quad \begin{bmatrix} 0 & 0 & 0.25 & 0.5 & 0.75 & 1 & 1 \end{bmatrix}$$
$$k = 3 \quad \begin{bmatrix} 0 & 0 & 0 & 0.33 & 0.66 & 1 & 1 & 1 \end{bmatrix}$$
$$k = 4 \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0.5 & 1 & 1 & 1 & 1 \end{bmatrix}$$

A uniform knot vector that is open can be expressed formally as $x_i = 0$ for $1 \leq i \leq k$, $x_i = i - k$ for $k+1 \leq i \leq n+1$ and $x_i = n - k + 2$ for $n+2 \leq i \leq n+k+1$.

The basis of B-spline reduces to the Bernstein basis when an open uniform knot vector is used and the number of control polygon vertices matches the order of the B-spline basis. The knot vector in the scenario is simply made up of k zeros and k ones [2]. For instance, the order $k = 4$ open uniform knot vector for four polygon vertices is $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$.

4. Numerical Example and its Visualization

In this section will show the numerical calculation of B-spline curve interpolation with open uniform knot vectors and their visualization for each PFBSCI membership functions consist of positive, neutral, negative and refusal membership. This numerical example at random and will make use of the interpolation technique. A picture fuzzy B-spline control point relation (PFBSCPR) will be shown in Table 1 that consists of 6 control point which is consider as x value with degree $k = 5$.

Table 1
Picture fuzzy B-spline control point relation with its membership degree

PFBSCPR $\widehat{C}_i$	Positive Membership μ	Neutral Membership η	Negative Membership ν	Refusal Membership μ
1	0.2	0.3	0.4	0.1
0	0.1	0.2	0.3	0.4
2	0.3	0	0.2	0.5
2	0.4	0.3	0.1	0.2
3	0	0.7	0.3	0
5	0.5	0.3	0	0.2

An open uniform knot vector starts with $k+1$ zeros and ends with $k+1$ ones. The middle knots are uniformly spaced. Hence, the knot vector U will be

$$U=\left[0,0,0,0,\frac{1}{3},\frac{2}{3},1,1,1,1\right]$$

From Eq. (8), U and Eq. (16) we will obtain the basis function matrix W, which is consist of 4 membership where W^{μ}, W^{η}, W^{ν} and W^{ρ} is denoted as basis function matrix for positive, neutral, negative and refusal membership respectively.

$$\left[W^{\mu}\right]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.1389 & 0.5965 & 0.2459 & 0.0187 & 0 & 0 \\ 0 & 0.0424 & 0.4978 & 0.4376 & 0.0222 & 0 \\ 0 & 0.0323 & 0.4719 & 0.4656 & 0.0302 & 0 \\ 0 & 0 & 0.1612 & 0.5804 & 0.2584 & 1.34\times10^{-6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[W^{\eta}\right]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.1499 & 0.5977 & 0.2352 & 0.0172 & 0 & 0 \\ 0 & 0.0523 & 0.5177 & 0.4132 & 0.0168 & 0 \\ 0 & 0.0234 & 0.4420 & 0.4939 & 0.0407 & 0 \\ 0 & 0 & 0.1435 & 0.5683 & 0.2880 & 1.15\times10^{-4} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[W^{\nu}\right]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.1324 & 0.5954 & 0.2525 & 0.0196 & 0 & 0 \\ 0 & 0.0378 & 0.4868 & 0.4499 & 0.0255 & 0 \\ 0 & 0.0284 & 0.4597 & 0.4776 & 0.0343 & 0 \\ 0 & 0 & 0.1600 & 0.5798 & 0.2602 & 2.42\times10^{-6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[W^{\rho}\right]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.1317 & 0.5952 & 0.2533 & 0.0198 & 0 & 0 \\ 0 & 0.0455 & 0.5044 & 0.4298 & 0.0203 & 0 \\ 0 & 0.0193 & 0.4250 & 0.5083 & 0.0474 & 0 \\ 0 & 0 & 0.1410 & 0.5662 & 0.2927 & 1.6\times10^{-4} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

As result, PFBSCI can be acquired by using Eq. (17) as below

$$\widehat{B}^{\mu}=\left[\langle 1,0.2\rangle,\langle -1.7319,0.9893\rangle,\langle 3.6237,-2.3822\rangle,\langle 0.1596,3.6392\rangle,\langle 8.9922,-1.6121\rangle,\langle 5,0.5\rangle\right]$$
$$\widehat{B}^{\eta}=\left[\langle 1,0.3\rangle,\langle -1.6426,1.0160\rangle,\langle 3.5138,-2.1115\rangle,\langle 0.3120,2.5819\rangle,\langle 8.0310,-1.6121\rangle,\langle 5,0.3\rangle\right]$$
$$\widehat{B}^{\nu}=\left[\langle 1,0.4\rangle,\langle -1.7861,-0.7046\rangle,\langle 3.6787,2.8678\rangle,\langle 0.1033,-2.9352\rangle,\langle 9.0369,5.9291\rangle,\langle 5,0\rangle\right]$$
$$\widehat{B}^{\rho}=\left[\langle 1,0.1\rangle,\langle -1.7502,-0.4530\rangle,\langle 3.5723,2.7605\rangle,\langle 0.2670,-2.1637\rangle,\langle 8.0106,2.8563\rangle,\langle 5,0.2\rangle\right]$$

where the four PFBSCI for $\widehat{B}^{\mu}$, $\widehat{B}^{\eta}$, $\widehat{B}^{\nu}$ and $\widehat{B}^{\rho}$ are denoted as positive, neutral, negative and refusal membership respectively.

Based on PFBSCI calculation as above, we visualized the curve separately in Figure 1, Figure 2, Figure 3 and Figure 4 with their respective picture control points (red dots) by using Eq. (6). Figure 1 until Figure 4 also known as positive (green curve), neutral (black curve), negative (pink curve) and refusal (blue) membership B-spline curve interpolation. To determine the points or degree of each membership, it determined through characteristic of PFS, which is followed from Eq. (2).

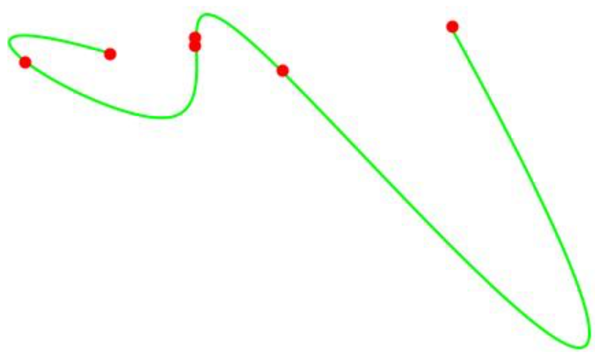

Fig. 1. Positive membership PFBSCI

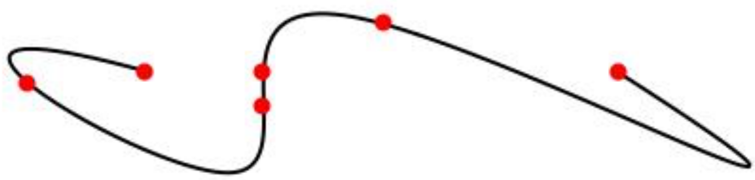

Fig. 2. Neutral membership PFBSCI

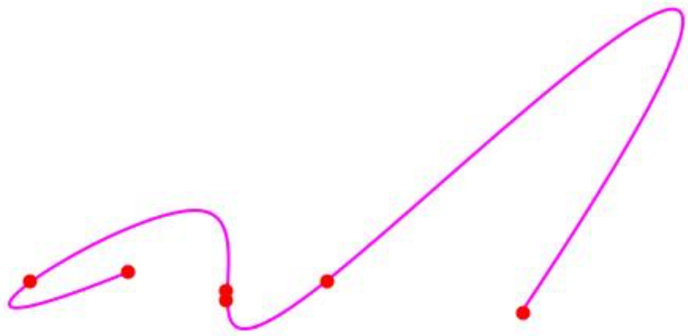

Fig. 3. Negative membership PFBSCI

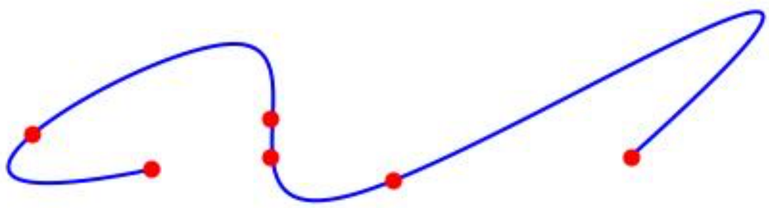

Fig. 4. Refusal membership PFBSCI

4. Conclusions

In this study introduced a novel method for generating picture fuzzy B-spline curve interpolation with an open uniform knot vector. Our method leverages the inherent flexibility of B-spline curves interpolation combined with picture fuzzy sets to create smooth and adaptive curves interpolation that can handle uncertainty and partial membership values effectively. By incorporating picture fuzzy sets with an additional degree of indeterminacy, this research has enhanced the ability of B-spline curves to model and interpolate data points with varying levels of uncertainty and hesitation. This research use of an open knot vector ensures that the generated B-spline curves maintain their end conditions, providing smooth transitions at the boundaries and preserving the shape of the control polygon. The proposed method has a wide range of potential applications, including computer graphics, computer-aided geometric design, and data visualization. By handling uncertainty and partial information more effectively, PFBSCI can improve the accuracy and reliability of these applications. Future research could focus on extending this approach to higher-dimensional data and exploring the integration of other advanced fuzzy set theories. This innovative approach not only addresses the limitations of traditional B-spline curves in handling uncertain data but also opens up new avenues for research and application in various field. Additionally, practical implementations and real-world case studies will be essential to validate the effectiveness and utility of the proposed method.

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